THE ART OF TEACHING ARITHMETIC

The Art of Teaching Arithmetic

A BOOK FOR CLASS TEACHERS

BY

Davies

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PREFACE TO SECOND EDITION

ANY reader of the following pages will quickly realise that I have written from the point of view of the Primary school. I believe, however, that the ideas here embodied are more generally applicable, the nature of the application depending on the social environment of the pupil. Hence teachers of Arithmetic in Secondary schools may find that parts of this book bear on their work. I have not written for Mathematical specialists.

Another limitation must be mentioned. I have accepted the present system of teaching children in large classes, not because I consider this ideal, but because it seems worth while to attempt the improvement in one direction of a state of matters likely to continue yet awhile. Some drastic social reconstruction seems necessary, both within and without the Primary school. One is glad to note that Doctor Montessori and the American and English advocates of "free work" even in Secondary schools preparing for external examinations, have recently given a much-needed fillip to "individual" as opposed to "class" teaching.

J. B. T.

Manchester,
January 1921.

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PART I

CHAPTER I

THINGS AS THEY ARE

To be asked to take a lower seat is always a trying experience: such has recently been the fate of the three R's. We have evening classes, vacation courses and special lectures in Drawing, Handwork, Nature Study, Geography and Physical Exercise, while meantime the daily bread of our school curriculum often groweth stale. As a result, we have tended to forget that the teaching of these subjects, too, may be made a fine art, that here also there is scope for all a teacher has of enthusiasm and originality, that even Arithmetic may become a happy pasture-land for our children's hungry minds and energetic impulses. If there should be anything of divine economy in the order of this world, surely it would be sheer waste if what is a necessity could not also be a good. We need, many of us, to learn how to make virtues of our necessities, how to transform what must be into what should be. As a first step, it may be wise to look straight into the face of things as they are.

Things as they are have, however, many faces: let us begin with the class-teacher's point of view. First, then, we have to teach Arithmetic and to teach it with appalling regularity, whether we like it or not: often, frankly, we do not like it at all. The subject itself seems dry to

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many, but apart from that, think of the apparent hopelessness of our task! We find ourselves with from forty to sixty children at different stages, working the primary operations in differing ways, often with no proper grounding. Standard II. and III. teachers may have many children who still add by ones; Standard IV. and V. teachers receive classes painfully slow and inaccurate in tables and in simple money operations; Standard VI. and VII. teachers have pupils who know a little of everything, but have a firm grip of nothing. Often the children come to us already bored or disheartened.

This is all bad enough, but the strain becomes serious when we consider not only our material, but the results demanded from us with such material. Frequently we are expected to cover a syllabus too extensive for thorough treatment by any of the class, certainly too extensive for the weaker part of the class on any reasonable method. On such a syllabus we have to get decent results in terminal examinations and to produce at least the appearance of successful mental work. We have to secure tidy books with the majority of the sums correctly worked, in order to prove to the inspector that the Arithmetic in our standard is satisfactory. In the upper classes we have to cater for the special instruction of any who by extra teaching may secure those scholarships which bring higher education to our pupils and a good name to our school. If we fail in attaining these results, promotion will be slow in coming or will never come at all, and promotion means so much when the salaries of Elementary teachers are as they are. Behind all this strain and stress the thoughtful teacher has to answer somehow his own questionings. What ought we to do? How far ought we to aim at satisfying our heads and our inspectors? What is really best for our children? What will equip them for life as good citizens?

The amount of effort put forth in teaching Arithmetic throughout the country must be enormous: wisely husbanded and used for clearly defined ends it could accomplish much. Some there are, a growing number, who are escaping from this prison-house of difficulty and drudgery

We see many who enjoy learning Arithmetic, and many who teach it happily even to classes of sixty. It is not that all the problems of this complicated task have been solved, but some of them are being tackled satisfactorily, and even in teaching Arithmetic nothing succeeds like a little success.

The variety of the teacher's experience is, however, nothing compared with the variety of the children's, for in our classes we find every type of temperament with every quality of intellectual endowment. There is the quick, lively child who thoroughly appreciates Arithmetic if he is allowed plenty to do, a condition by no means always fulfilled. Such a child, sensibly taught, takes pride in solving problems and finds a pleasurable excitement in getting sums right, however mechanical the drill may be; he may taste for himself the joy of work for work's sake, in certain cases reaching even an æsthetic satisfaction in the accomplishment of his task. Given too little to do, he loses interest and becomes a nuisance to the teacher who believes in his class working at the average speed.

At the other extreme we have the slow, dull child, born a low-grade intellect or rendered such by lack of food, sleep, and fresh air, with its resulting lack of vitality. To such children Arithmetic must always be hard because of its demand for effort and concentration: as things are it is often almost unbearable. Day after day it returns: day after day there is the hopeless effort to understand what is wanted of them, the depressing experience of scarcely over working a sum correctly, and in many a case the horribly unjust fate of being punished for failing to do the impossible. The more such children are scolded or caned or "banged about," the more slow and dull do they become, while in the case of nervous temperaments the far-reaching consequences may be appalling. The trouble is that most of us find it so difficult to make our imaginations realise the true condition of the weak and downtrodden, especially when their presence is a nuisance to ourselves and a drag on their more fortunate brothers and sisters.

Between these two extremes lies a variety of types with mediocre Arithmetical powers. In some classes we find these children filled with eagerness in improving the old work and with zest in tackling the new, developing their ability to its utmost. In many a class these children are stifled by the monotony of their daily Arithmetic lessons, or are suffering because their interest and effort are aroused, not by the presentation of the subject in a naturally attractive form, but by fear of punishment. Without doubt from such children decently good results can be secured by fear, but is the good Arithmetic worth the cost to the child? That is another staggering question to the thoughtful teacher. Meantime not a few of even the thoughtful amongst us deliberately by force obtain respectable results from this central portion of their class. What can be the attitude to Arithmetic of children who learn it under such a régime?

The whole situation is more complicated because we cannot wholeheartedly condemn even heads of schools! They, too, have their own problem with its peculiar difficulties. We have, of course, in increasing numbers, the up-to-date head master who has his eye on the child and on the State rather than on any inspector or any scholarship examination. But, even then, is his task an easy one? The needs of the child and the demands of the State are hard to meet. Not all his teachers are fit or willing to carry out his cherished plans and designs. He wishes to do his best by his pupils, and so must prepare them for a scholarship examination which requires the ability on one fixed day in the year to solve problems based on a wide and varied syllabus. Some inspectors, too, judge superficially and advise rashly, and though the old days of payment by results are over, inspectors' reports do appear in local newspapers. It takes a strong man patiently to bide his time for good results which may be slow in coming, while others less particular about their methods and more amenable to outside pressure carry off glory and chances of promotion. Such men know well the complications of the problem of teaching Arithmetic successfully. They would welcome help from any quarter where the difficulties of things as they are were frankly admitted. Meantime they regret that educational conferences and summer schools bestow so little attention on Arithmetic.

Here and there we find, of course, the head whose methods are comparable to those of a slave-driver. Results, in the narrowest sense of the word, are what he wants, and to secure them he does not hesitate to coerce his staff and bully his pupils. Even such a man adopts a reasonable position. It is a head master's business to obtain good results: teachers being too soft or too slack require a strong driving-force behind them: children being lazy and careless must for their own good be compelled to work.

Between these two extremes we have again a great variety. Many men and women have had their own early experience in the days of payment by results, when it was absolutely necessary to secure the appearance of "good discipline" and, by hook or crook, to cover a definite syllabus. Times have changed, but such find it hard to realise that better work can be done in a freer atmosphere, and that the hum of natural activity is not the same as noise and bad order. As they generally know that Arithmetic is unsatisfactory, they look back with longing to the days when children at least learnt their tables properly and developed respectable speed and accuracy within narrow limits. Lacking a clear vision of a complex situation, they have lost sight of the true proportion of things: confused and troubled, they tend to be moved by every wind that blows. These pull and push and spend an enormous amount of effort, feeling all the time that things go far from well. Enduring the burden and heat of the day, they often grow weary and discouraged.

Not only within the school is there more or less conscious criticism of things as they are, but outside also the matter is under discussion. We may do as we like with Geography or Nature Study or Drawing, but parents expect their children to be able to "figure" correctly. Their requirements are, however, seldom stringent, and in the majority of cases it is probable that the schools sufficiently satisfy them. On the other hand, business men and employers of labour are very critical indeed of the present product of our Arithmetic teaching. Their demand is definite: they want speed, accuracy, and, if they can have it as an extra, common sense. Their contention is that in school we teach much that is of little value, and leave untaught the things which matter most. Their frankly expressed opinion is that Arithmetic is going from bad to worse.

From all this medley of impressions, we may repeat a few statements which are commonplaces in our discussion. Arithmetic is an important subject in our school curriculum, a certain amount of it being an absolute necessity to the child. Because of the effort it requires, it is a hard subject for children to learn: because of many circumstances, it is a difficult thing to teach. Meantime we are in a transition period: the swing of the pendulum has carried some of us farther than we know, or, to change our metaphor, we have left one safe haven to find a better, and some of us have lost our bearings. Others have steered more wisely, having glimpsed a guiding light. To such wise men and women this book is due. It is written for ordinary class-teachers, with many children to handle and difficult situations to tackle, in the hope that a little more light may be theirs, whereby each may, in his own vessel and by his own route, make for a better anchorage.

CHAPTER II

THE NATURE OF ARITHMETIC

WHAT do we think of teaching when we speak of Arithmetic? What does the word convey to our pupils and their parents? The most ready replies to these questions are that Arithmetic consists in working sums, or in counting, or in dealing with numbers, or in handling figures, or in going through the exercises contained in an Arithmetic book. This is the superficial impression usually produced by work in this subject.

But the truth about the nature of Arithmetic is something far more thrilling in its human interest. After all, we cannot use figures or numbers apart from things which matter to us: in real life there is always some cause for our calculations. Let us look back, then, to the early days of our race and see if we can discover why we human beings learnt to count. Long, long ago, as the stories put it, men lived much more simply than we live now. They had the wide spaces of the earth to roam over: only the setting of the sun put an end to the freedom of their activities: they could eat of the fruits of the earth or of the abundance of animals which could be killed. Seldom were the limitations of food or of space or of time realised by them. But as men began to know that they could not have all they wished for, considerations of "mine" and "thine" crept in; men were compelled to keep count of their flocks, compelled in some way to measure their time and their goods. The more complex life became, the more had men to attend to the quantity of things as well as their quality. Learning to count was a distinct stage in the development of our race, a stage which even yet all have not taken; for instance, they tell us that in the Malay Peninsula are savages who cannot count beyond three.

The development of a little child is on similar lines. At first the room or house is to him a spacious abode: he is conscious of no time-limit and his food is always there. When he comes to want things for himself, to find that he can have only his own share of the good things provided, to realise that not at every moment can he go out or be fed, he, too, begins to feel the need of counting and measuring. Very often he begins by counting sweets, because just here is he most definitely limited. There is some sense in learning the difference between two chocolates and three or maybe four!

Is not the same thing true of ourselves? We desire some definite end: it may be the provision of food, the fashioning of a garment, or time for some special study. We possess means to help us to the attainment of that end—a fraction of our salary, a piece of cloth, part of our summer holiday. The more we feel the insufficiency of the means at our disposal and the more puzzled we are as to how to adapt these to secure the desired end, the more Arithmetic we shall have to do. The less our salary, the more do money calculations dog our footsteps: the shorter our cloth, the more accurate must our measurements be if we would produce the garment we desire: the greater the demands on our time, the more carefully must that time be planned. It is our limitations which compel us to a quantitative study of the universe. Out of the adaptation of means to a desired end does the science of Arithmetic spring.

But men were not long in discovering that this measuring activity evolved in us by our limitations has bestowed upon us powers beyond our expectation. We were compelled to create a new instrument which in itself is a valuable possession. The child who has learnt to tell the time finds that his life is his to order and arrange as never before. The woman whom scarcity of cloth has forced to careful measurement will not again cut out by guesswork, because she will have learnt from her experience that a little Arithmetic enables her to produce more shapely garments. Again, it is only as men have advanced in the quantitative study of the universe that the progress of civilisation has become rapid. When rough methods of barter had led gradually to a well-defined system of money-exchange, commerce came into her own. For many centuries men had known of magnetic and electrical phenomena in connection with thunderstorms, amber and the compass needle, but not until after 1780, when the first mathematical law of the subject was formulated, did the science of electricity affect the development of civilisation. Before 1780, how slow the progress: since 1780, how amazingly rapid! Thus we see that learning to count and to measure gives us power over ourselves and power over our environment.

To sum up, we may state that Arithmetic is the study of the quantitative aspect of the universe. The idea of quantity enters into all the activities of life, because of the limits imposed on us by the universe. Primitive man and the child, because they are scarcely conscious of these limits, require little Arithmetic. And on the other hand, after the study of the universe became quantitative, more rapid progress in invention and civilisation was possible.

What, then, should be our attitude to this subject which we teach? Are we to regard it as something fixed and rigid which we of necessity have learnt, and now, equally of necessity, must teach? Is it not something much greater? It is no fixed material to be thrust upon our children, but a gift of the race to children, a gift found to be of such value that not one of our pupils can afford to lose his share in the blessing. Arithmetic should be regarded as a developing race-product, knowledge acquired by man as he "grew up" into civilisation. Some things which our ancestors learnt we need not learn. For instance, the attempt to grasp the method of long division common in the fifteenth and sixteenth centuries would give many of us a bad half-hour. Again, certain business applications of the fourteenth century, which were long taught as Arithmetic, have at last disappeared from our syllabuses. The next great change may well be the entire abolition of our heterogeneous system of money, weights and measures, and the introduction in their stead of the simplicities of the metric system, a change which would revolutionise the work in Standards II., III. and IV.

Before leaving this conception of Arithmetic as a race-product, it may be worth while to point out that from the study of the historical development of Arithmetic we may gain some guidance in our teaching. As there is some degree of parallelism between the development of the race and the development of the child, we should, if we understood when and how the race learnt some branch of Arithmetic, probably be helped to discover when and how we could most successfully teach that branch to our children.

CHAPTER III

WHY DO WE FAIL?

To many of us this description of Arithmetic as a good gift of the race to children must seem like irony, when we recollect how frequently our pupils dislike Arithmetic and how seldom results are satisfactory. Moreover, the greatest danger of the present state of affairs is that we are encouraging in our children those fatal habits of listlessness and inattention which must diminish their thinking powers for life. Considering the number of children daily receiving Arithmetic lessons, we may say, without exaggeration, that through this waste of mental power the nation suffers heavily on its intellectual side. In this chapter we propose to discuss some of the causes of this serious failure.

Much of the trouble is due to the lack of righteousness in our social order: behind everything lies the "social problem." Nothing can possibly justify the housing conditions under which many children are forced to live, or the low wages on which healthy family life is an impossibility, or the employment of children on heavy work out of school hours. Again, until the nation counts money spent on its children a sound investment for the future, we shall continue to teach classes which are much too large in badly ventilated rooms; we shall go on wasting money on the education of children physically unfit without spending more than a mere pittance in the attempt to combat physical distress. Meantime we teachers have before us the half-fed, half-slept, sickly children already weary with other work, and we are well aware that with such handicaps no teaching of ours is likely to be much of a success. Arithmetic, because of its special demand for effort and concentration, suffers severely. Here the call is for a clear-eyed appreciation of the odds against us, and for courage and self-sacrifice in facing them.

Part of the blame for the present state of Arithmetic rests with those set in authority over us, be they representatives of the central and local authorities, school managers, or our head masters. Not all inspectors are wise men, while because of their powers over us, their failings press heavily. Again, those in authority give or withhold apparatus: good blackboard accommodation, simple mathematical instruments, apparatus for duplicating and a plentiful supply of paper of various kinds, make a vast difference to the possibilities of our lessons. On the other hand, complaints are sometimes justly made that our demands are immoderate and our methods uneconomical. Simplicity and moderation in our requisitions, coupled with a careful avoidance of waste, are of the essence of wisdom in this matter.

Our heads, of course, more directly influence our teaching. They arrange our time-tables, and we may rebel against unduly long Arithmetic lessons or the necessity of taking practical and oral work at fixed times, of which more will be said later. The kind of tests which our heads set, along with the syllabuses which they make, limit our freedom as regards treatment of the subject; while their insistence on methods with which we disagree, or their demands for orderly silent classes, or their emphasis on beautifully kept notebooks are pricks against which not a few of us must kick. Those things are serious, but we would often make more headway against them if we frankly recognised that the teaching of Arithmetic is a fine art never approaching perfection and hence always open to discussion. Such an attitude would make us less self-conscious about our teaching, more willing to air our real difficulties to inspectors and heads, and more sincere in seeking their assistance. Often simplicity and frankness will result in a good compromise, though occasionally a time may come when we have to refuse compromise and take the consequences.

Needless to say, tiresome parents and naughty children interfere with our efforts to teach Arithmetic satisfactorily. The difficulty with parents is seldom serious, because pressure can now be brought to bear on them to secure regular attendance and the healthiest possible physical conditions. Naughty children are always with us. Because of the immaturity of the child, or because of his bad upbringing, or because of his vicious environment, his standards of morality may be low, and often we have to combat the lack of good will or the positively bad will. In general, however, we make far too much of the naughtiness of our pupils, counting as evil what is but their natural reaction against the deficiencies of our teaching. At times we shall have to fall back on "must": having failed to woo, we shall have to force bad wills to work. Only let us take heed lest we blame our pupils for the fruits of our own ignorance or negligence. All of this has a special bearing on Arithmetic teaching because of its emphasis on results.

We come now to the case against ourselves, to those causes of failure directly due to class-teachers. In the points already raised the blame belongs to others: we may or may not make the best of a bad situation. Here we have to judge ourselves. For instance, one definite cause of failure is that we pay so little regard to our aims in teaching Arithmetic; we do not state definitely to ourselves what we intend to accomplish, or we lack a clear vision of the nature of the results to be desired. This is so important that we shall devote the next chapter to a study of our goal in Arithmetic lessons.

We realised in considering the nature of our subject that Arithmetic has to do with human life, and sprang out of human needs: often we fail because we do not link up our teaching with the lives and needs of our children. Some motive we must provide, and the best and simplest is, as far as possible, to join our number work to the natural instincts and desires of the children. Here come in the value of imagination and the wisdom of intercourse with child-friends outside school. We know how some teachers instinctively put their problems into attractive form, while, in the hands of others, the same work seems formal and dull. This is partly the result of a difference in imaginative endowment, and partly the result of lack of knowledge of children's interests, as seen in a less artificial setting than is provided by the majority of classrooms. We can learn to word our problems so that they appeal to our pupils: town and country children, the well-to-do and the poor, boys and girls, children of various ages, all will require different types. What may have served us well on one occasion will have to take new colour to suit another set of children. Variety is essential. Again, at times we can by dramatic games link Arithmetic with child-life. The love of playing shop is ingrained in all: much work in money, weights, and measures can be done in the lowest standards in connection with a shopping game. A Standard I. class for several weeks centred their Arithmetic round a soldiers' camp: tents holding ten men were the ten units, groups of ten tents the hundred units. The fundamental operations in simple forms arose naturally out of the practical problems of feeding the camp, sending dispatches, and carrying through drill and manœuvres. Throughout Arithmetic use can be made occasionally of the dramatic instinct in our pupils. To suggest other possibilities, games often involve us in calculations; problems may arise or be invented in connection with some story read by the children in another lesson; sometimes we can utilise an interest in gardening or needlework or drawing; children can invent problems for themselves and for each other. One small boy, for instance, was very anxious to discover for how long he could use his electric torch, guaranteed to give him 1000 flashes. These are mere suggestions, thrown out at random to indicate the possibility of collecting, each on his own lines, material which will help to ensure children's interest and concentration.

Some of us fail, however, on the more technical side of Arithmetic. Often we have ourselves been taught mechanically, and, until we come actually to teach, do not realise how little we understand our work. How many of us, before our teaching experience, understood clearly the meaning of our decimal operations, the foundations of the various subtraction methods, the reasons for the remainder rule in division by factors or for the division rule in vulgar factions, why there are twenty-eight pounds in a quarter, or even why "hundredweight" should be written "cwt."? It is in the belief that many teachers need help in these or similar difficulties that Part II. of this book is written. Again, when we do for ourselves understand what we have to teach, our mastery of it may not be sufficiently thorough to enable us to break down the whole operation into a series of steps, each of which can be surmounted by our children without undue effort. The result is that we are tempted to teach a rule mechanically or demonstrate its explanation, instead of giving our pupils the training and thrill of being explorers. No child is likely to invent the method for practice sums of any difficulty or to discover for himself the rule for division of fractions. But if each of these is subdivided into a series of graded steps, many children can take each step for themselves without direct assistance from the teacher. Here, again, is need for knowledge which it is hoped Part II. may supply.

Akin to this is the fact that we tend to become too dependent on the text-books provided in school, and so allow slackness over preparation to creep in. No text-book can suit any one class to perfection: the best of books needs to be used with discrimination and adapted to the special weaknesses of a particular class. It is so easy to work straight through such books—another reason for the monotony of many Arithmetic courses. We need to rearrange, select and supplement if we would secure the best possible results: or, stated from another standpoint, our detailed schemes and syllabuses require much more careful planning than they often receive. Again, even working lessons require thorough preparation. Let us take a very humdrum example. A class does multiplication of money by factors badly. The reasons of the bad results may be various. Multiplication tables may be weak, in which case difficulty will arise in finding factors and in the actual multiplication. Here drill in some one table (*e. g.* 7), and a working lesson giving practice in the use of it (*e. g.* multiply money by 21, 35, 70), will lead at once to a marked improvement. Or the weakness may be in changing pence to shillings or shillings to pounds, in which case what is required is probably oral drill in these weak points, supplemented by some definite teaching with regard to them: if we fall back on the pence table, this teaching will take one form; if on the twelve times table, it will take another form. Again, the weakness may be in understanding the idea which underlies the factor method of multiplication; then simple examples, fully stated, in connection with a variety of problems, will be necessary. One part of the class may need one kind of special help, another part another kind. It is little good working vaguely to improve weak multiplication by factors. To effect a cure we must always diagnose and provide the specific remedy, all of which demands regular preparation. To go to an Arithmetic lesson without knowing exactly what examples are to be used and the nature of the difficulties likely to arise from them is seldom justifiable. Closely connected with this is the tendency inherent in some of us to shirk practical or oral work because of the extra strain and preparation which they involve, a tendency which has resulted in the common practice of definitely allotting certain periods to such work. We postpone this difficult point to a general discussion of practical and oral work, which will be taken in Part III.

Probably such a scientific treatment of children's weakness in Arithmetic as has been suggested in the preceding paragraph would of itself do much to alter one common cause of failure—our attitude to errors in Arithmetic. For some reason, when we "grown-ups" make mistakes in calculation, we expect pity rather than blame; but the same grown-ups will scold or punish children for similar faults. It is not that we are to condone mistakes: faults in Arithmetic in general must be corrected and the value of accuracy insisted on. At times, too, mistakes certainly due to moral failure will have to be punished. But the all-too-common practice of speaking crossly to children who make mistakes due to our bad teaching, or to their physical unfitness, or, in some cases, to real intellectual weakness, simply defeats its own end, even where the evil is not increased by caning for sums which are wrong. Working alongside of our children, we are striving to produce in them powers which will lead to accurate and sensible results: this should be our attitude rather than the expectation or demand that sums shall be right. To punish for mistakes is, in general, a relic of barbarism. The only suitable punishment for making mistakes is to have to correct them: the justice of this is evident.

This, however, is but part of a larger problem. Our success or failure in teaching Arithmetic depends far more than we realise on the kind of atmosphere we create in our classrooms. The real difficulties of Arithmetic need to be lightened by an abundance of smiles and laughter; patience, sympathy, kindness, and justice are essential, while the zest and interest of the teacher must be sufficiently obvious to induce similar feelings in his pupils. A keen teacher means ultimately a keen class. We need to study individuals, expecting from each only what is possible, wisely discriminating in our praise and blame. (A backward child working two sums correctly may deserve more praise than a quick child working six!) Such intimate intercourse with a large class and such real goodness in ourselves are, of course, hardest of all to secure. Some of us will fight the battle in teaching other subjects more congenial to us, carrying over to Arithmetic the fruits of our success elsewhere. For most of us it is probably true that this battle has to be won before a lesson begins, not after.

Let us summarise briefly on the positive side the results of this self-criticism. We must study the underlying ideas and processes of Arithmetic, study the mind of the child, and study the possibilities of our syllabus. We must connect Arithmetic with the actual life of the child, training ourselves to watch for suitable illustrations and problems. We must go on making careful preparation for our lessons, refusing to be satisfied with previous good results. We have to consider most carefully our attitude to children's mistakes, avoiding on the one hand harshness and injustice, and on the other hand slackness and habits of inaccuracy. We have, finally, to create gradually in our classroom, by patience, sympathy, zest and good-humour, the kind of atmosphere in which a child can do his best. Of which things not one can be accomplished in a day!

CHAPTER IV

OUR AIM

We have already referred to the common confusion of thought concerning our reasons for teaching Arithmetic and concerning the nature of the results desired. Trifling are the motives often in the foreground. Pleasing our head masters, teaching Arithmetic so well that frequently class and teacher will experience that thrill which is the product of the lesson characterised as "alive," getting good results in examinations—all these may be worthy enough motives, but are, nevertheless, trifling. In the struggle of educational effort we need a certain detachment which belongs only to those who have scaled the heights, looked forth over the wide landscape, and have then brought down to the petty details of their work the steadiness and inspiration of a definite, wisely chosen purpose.

When we set ourselves to meditation on this matter we find two apparently distinct purposes which may animate our efforts. Our eyes may be on the economic struggle into which our pupils must so soon be precipitated, on the details of the lives which will be theirs, on their future citizenship. Then we may state our purpose to be the teaching of that in Arithmetic which will be of use to our pupils, of that which will fit them to take their place in the society to which they belong. We may thus regard our work from the utilitarian point of view, taking "utilitarian" in the narrow sense: as a background to this may be fitted in high desires of social service to our pupils. Others of us may have our eyes on the more academic aspect of the matter, our purpose being to teach children to think. Whatever may be the exact meaning given to the word "think," we obviously aim at some kind of intellectual training which we feel that we ourselves received from Arithmetic: we wish to perfect as far as may be the mental powers of our children. Clearly these are not two distinct purposes. It is useful, even in the narrow sense, to be able to think; our business men would like common sense as well as speed and accuracy. Nevertheless, although we need not regard the two as in watertight compartments, they do indicate a useful subdivision for our discussion.

At this stage it is worth while to make a note on the meaning given to the term "Arithmetic" in this book. It would be more correct to use "Elementary Mathematics," but as the greater part of the Mathematics done in primary schools has to do with number work, and as the Algebra and Geometry we do introduce is closely connected with number, we retain the common appellation of our timetables. In these, "Arithmetic" is not likely soon to be ousted by "Mathematics." Let it be granted, then, that in using the term "Arithmetic" we include some Geometry and do not exclude the possibility of Elementary Algebra.

First, we ask ourselves what it is strictly necessary we shall teach to our pupils in Arithmetic. To begin with, our children do not understand the value of any simple number, still less the meaning of notation whereby we can express clearly numbers too great otherwise to be grasped. They have at first no idea of two or seven, still less of twenty-five or three hundred and sixty-nine. This is not the place to discuss in detail how these ideas do become part of our children's mental outfit, but one thing is obvious: while they are themselves abstract notions of the mind, they arise only in connection with experience of concrete realities. Therefore we have first of all to give the ideas a chance to develop by providing a sufficient variety of concrete experience, and then gradually to make the children independent of such concrete aids so that they may reach a higher stage of mental development. What is true of the ideas of numbers is true also of the ideas of the fundamental operations, of addition, subtraction, multiplication and division, including the notion of a fraction. These are ideas of the mind, but they can be formed only in connection with concrete experience. Addition is a mental operation, but we do not form any abstract idea of it until we have many times put real things together with some definite purpose to achieve by putting them together. The same is true of all. Our children need to have much play-work with concrete apparatus, games, handwork and stories, so that these ideas may have an opportunity of developing; but they are not part and parcel of a child's mental life until any real problem calls up the appropriate mental exertion without the operation being of necessity performed concretely. How much labour the work outlined in this paragraph requires, only the teachers of young children know. But on it, as a foundation, rests the whole superstructure. Consider, for instance, what lies behind the working of any problem. When we adults work an Arithmetic problem, the realisation of the concrete situation which it embodies evokes from our minds the appropriate mental exertion: when our minds have abstractly, or with the visual aid of the symbolic figures, reached an abstract result, we transmute our abstract result into terms of the concrete problem requiring a solution. To be able to do Arithmetic means the possession of a mental instrument for dealing with material problems. So our children must learn to pass without difficulty from concrete to abstract and from abstract to concrete. The necessity for strength and solidity here demands that, in the earliest stages of number work, advance shall be slow and sure.

But it is one thing to have the concrete experience evoking in our pupils the appropriate mental work to deal with it, and another to have this mental work performed quickly and accurately. The importance of this point is due to the fact that on speed and accuracy in the use of the fundamental operations depend speed and accuracy in solving all more complex problems. Children who can quickly add, subtract, multiply and divide within easy limits, do grapple with harder matters with amazing ease. To fail here is to fail in all, and it is our failure here which mars much of our work and makes business men so bitter about the product of our present teaching. This, for the moment, is the most glaring weakness in our Arithmetic. We must therefore make time for drill work. It is probably true that from Standard II. to Standard VII. one lesson a week devoted to drill in work with abstract numbers, aiming only at the development of speed and accuracy, would quickly alter matters. Our books, once devoted almost entirely to mechanical Arithmetic, are now afraid of the least suspicion of catering for a necessity, a necessity, too, which is by no means an evil. Our children see sense in becoming quick and accurate, and find the exercise pleasurable and stimulating. At first competition between groups may be a help.

One other cause for our present failure here is that now, when our books contain so great a variety of examples, it is more difficult than in former times to set children to work by themselves. As a result, our quicker pupils seldom work up to the limits of their ability: so we lose our best. We must always provide sufficient work to permit of quick pupils developing in speed. This is a troublesome problem in class management which will be treated later in this book. Meantime let it be reiterated that, in order to prepare our children for life, we must aim definitely at speed and accuracy in the use of the fundamental operations.

In addition there are certain common applications of these operations to everyday life with which our pupils must become familiar. We have all at times to deal with calculations concerning money, weights and measures. Besides the fundamental ideas about fractions, we have to understand a little about percentages, used so commonly as a good alternative to ordinary fractions. Again, we all have something to do with mensuration, or, as it may be called, Geometry: we need to know how to deal in a simple way with areas and volumes in general, and in a more detailed way with the areas and volumes of certain common surfaces and solids. With certain business matters most people ought to be slightly acquainted: for instance, with interest, rates and taxes, insurance, investment of money. In all these cases what is required is a knowledge of new facts bearing on each situation:

this new knowledge has then to be used in connection with the ordinary fundamental operations. For example, learning avoirdupois weight means acquiring certain new knowledge about weighing, about different methods of comparing weights, about different standards for weighing and the connections between them. This done, or while this is being done, problems are set involving the application of addition or subtraction to this new knowledge. The real number work is not new. It is the old over and over again, but the material to which it is applied is new and is in such common use that special time is set aside for acquiring facility in dealing with it. Again, if once a child understands how money grows when it is used, how the idea of giving interest on money so used arises, how that interest is best measured, and in a simple manner how labour and capital are related, the actual solving of the necessary problems in connection with this is straightforward work in number. Exactly what applications shall be taught, each generation must decide for itself: what will be of value to one generation will not be of equal value to the next. The introduction of the metric system, as has already been pointed out, would revolutionise our syllabuses. Again, the applications will vary from district to district and from school to school. A girls' school will study in detail different applications from a boys' school: country children naturally have a slightly different syllabus from town children. But the majority of the applications usually taught in school are of common interest and, in general, necessary if our children are to deal sensibly with the details of their lives outside school.

But what is to be our attitude with regard to the special applications required for specific trades or businesses? Should the Elementary school teach that Arithmetic which employers may later demand from their employees? It is obvious that frequently children will leave school about the same time to go to such a variety of occupations that detailed preparation for all is an impossibility. Let us state briefly certain considerations. We cannot prepare all our pupils for every possible type of future work; moreover, the work they mean to take up may not be the work in which they will remain. What is demanded from us, if we aim at preparing them for their future lives, is that we shall have taught these applications common to all in such a way that our pupils shall be able independently to learn the special applications of the fundamental operations to their trade or business. To put it simply, a "sensible" boy or girl will be equal to any of the ordinary demands of their occupations. We must teach what applications we do study so as to produce sensible pupils. How much schools and teachers vary in their success in producing common sense in their pupils is incredible until one sees it for oneself. There are, however, certain occupations, such as engineering, which make excessive demands on a boy's Mathematical powers. In such cases it is the province of evening classes and technical schools to provide for the boys of the district such extra assistance as may be necessary: here, too, well-trained boys will find their bearings with comparative ease.

But sometimes even the quickest and most intelligent of our pupils may seem to fail when at first plunged into the vortex of daily work outside school. Here the cause probably is that they have been accustomed to do their Arithmetic only in a room where all are doing the same work in a studious atmosphere. They have been used to the help of peace and quietness and to the mental support of a class of fellow-workers in Arithmetic. It is a different matter to perform calculations on any scrap of paper, in any position, in the midst of a crowd engaged on other forms of work. As a sane preparation for conditions outside school, it is wise at times to have a variety of work carried on in the classroom, the law of silence being cancelled. Concentration under difficulties can never be learnt except by practice.

Summing up the results of the previous paragraphs, we may say that in order to fit our pupils to take their place in the society to which they belong, we must see to it that they understand the fundamental ideas of number, that they acquire speed and accuracy of calculation in using these, that they learn to apply their number ideas to the purposes of daily life, and, moreover, learn in such a way that there will be developed in them the power later to apply their knowledge to the special methods of their trade or business. This is a definite statement of part of our aim.

But some of us desire to give our pupils more than this from their Arithmetic lessons: we aim at teaching them to think. This, of course, is closely akin to the desire for common sense noted in a preceding paragraph, but there the emphasis was on the fact that common sense pays: here it is on our wish to make the very best of the mental endowment possessed by any child. It is wonderful to be a human being in possession of intellectual faculties; the talents bestowed must be used to their utmost capacity. We aim at developing that in our children which distinguishes them from the brute creation, because this development is for the good of each individual and for the good of the race. It is easy, however, outside the classroom, to indulge in such high-sounding idealism, but in the thick of the struggle our dreams of the ideal tend to fall away under the pressure of the actual. This is largely due not to overmuch dreaming, but to the poverty of our dreams. We have had but a vague vision of the end we desire, and no vision at all of the means for reaching that end. Could we see definitely the end, and even a little of the means whereby to attain it in a crowded classroom, our ideal would survive its contact with reality.

Let us first try clearly to state what is involved in our use of the word "think." Here examples will help Take two common experiences. A (a woman who thinks) and B (a woman who does not think) have each to make a blouse from a new pattern and spend for the first time a salary of £150. B will spend her money as she gets it, carelessly and impulsively, regardless of the need of saving to provide herself with a new winter coat or a good summer holiday. By the end of the year she will probably have run herself into debt simply because she did not think. Each year is likely to be as much of a muddle as the one before. B, again, will lay her new pattern on her cloth, one piece at a time, to discover at the end that she either has insufficient material or must place seams where no seams should be. The pattern probably will be of a size not large enough for her: the chances are that she will make no allowance for seams and will turn out too small a garment. As B never thinks, there is little hope that a second garment will be an improvement on the first. A's attitude to life is totally different. Spending £150 a year is to her a serious matter. She will make time to consider in advance her probable expenditure, working out a sensible forecast with an allowance for alternatives and emergencies. Even so, in the first year she is unlikely to be altogether successful: the unexpected happens. But afterwards she will more or less consciously analyse her year's expenditure, and, profiting by that experience, outline a more sensible plan for the next year. She is mistress of herself: B tends to become the slave of her impulses. A, using the new pattern, will compare it for size with an old, well-fitting garment. She will, before beginning to cut, plan how best the shapes can be taken from the given material; if extra seams must be, they will be in carefully selected places. The resulting blouse may not be a perfect fit, but a second one made by the same pattern is certain to be successful.

These cases might be multiplied indefinitely by any one of us. Here what is involved in the use of the word "think" appears to be forethought and subsequent analysis of results, or, in other words, before a new experience looking ahead and planning, and then after it, looking back and criticising with the desire to be more ready for any similar experience in the future. Notice how Arithmetic can afford frequent practice in just these things. We train our children before beginning a sum to plan how they are to work it; we discourage those impulsive beginnings which come to an end in three lines and produce a ragged page! The whole point is that we shall discourage them not because our head master dislikes ragged pages, but because it is bad preparation for life to encourage imprudence and impulsiveness. Every time that different methods of working or setting down a sum are placed on the blackboard, we can give our children training in analysing them in order to discover which is best. The same result, as far as future work goes, might be achieved by a dogmatic assertion that *this* is the way to do that sum, but the result in mental training is totally different. Children ought to discover for themselves why this is the best way. Good instances of this will be outlined in the sections on multiplication by partial products and on the teaching of practice. Again, before starting seriously to a piece of practical work, such as measuring the circumference of a circle or making a box, the children, after a few minutes of experiment, should be encouraged to formulate their various plans and to analyse the suggested plans so as to discover which is best. Those left may be two or three in number, and only the analysis of the finished product will help to decide the best plan of all. Other subjects, of course, afford similar training, and the whole life of children in school should be guided on these lines: it is useless to practise this mental attitude only in Arithmetic. But Arithmetic affords an excellent drilling-ground.

Take another class of instances. We are, many of us, given to making statements which are mere opinions cloaking our ignorance, opinions which, if analysed, might be transformed into true knowledge and so give us power to alter our surroundings. Here are two examples. We say, "I hate Mondays in school; they are never a success," or "Dorothy is a lazy child; she never puts forth effort." Let us take time to "think" about these things. Will Mondays always be bad days? Must Mondays always be bad? Why do I hate Mondays in school? Is it the fault of the school buildings, or of the children or of myself? Perhaps the buildings are insufficiently aired or heated; perhaps the children by 9 a.m. on Monday have forgotten what they knew at 4 p.m. on Friday, or have become more enamoured of open-air exercise, or are less docile; perhaps I have over-exerted myself during the week-end at work or play, or I shrink from the sordid school atmosphere after the beauty of nature, or physically I am not tuned up on Mondays to the pitch of energy needed for a full day of teaching. The true reason or reasons fixed, I may have power to surmount my failures on Mondays. I can bring pressure to bear on the caretaker: I can provide in advance for pupils' forgetfulness, and even to some extent for their lack of docility and effort: I can avoid overstrain at future week-ends, smile at myself over my natural preference for beauty, or arrange lessons so that Mondays shall not be days of heavy teaching. If Monday can never become the best day in the week, it may at least cease to be a failure. Thought, too, may prove that Dorothy's laziness is due to eye, ear, or throat trouble, or to lack of food or of sleep, or perhaps to real moral failure. The help of the doctor, or of school dinners and a talk to Dorothy's mother, or perhaps the help of some specially devised suitable-for-Dorothy punishment, will alter matters. In these instances "thinking" implies analysis of our opinion in order to discover the causes underlying it, that so we may gain power. In this analysis such questions as "Will it always be so? Must it always be so? Why?" will in general demand an answer.

The upper-standard work in Arithmetic often gives excellent practice in this kind of thought. Children obtaining slightly different results for the ratio of the circumference of a circle to its diameter almost invariably jump to the conclusion that the ratio varies because the circles are of different sizes. By asking them to suggest possible reasons for, say, Kate's silence, besides the obvious one of lack of friendliness (*e. g.* Kate may have a headache, may not have slept, may have trouble at home), they can be led to appreciate the fact that it is wise to look beyond the most obvious cause for the variation in the ratio. Other causes will then be suggested by the class—faults in the roundness of the objects, inexactness in measuring, slipping of string or of object, or even errors in division. The children will see that there must be inaccuracy in their results, and will be thus prepared for their teacher's dogmatic assertion that Mathematicians have proved that the ratio is an unvarying one and is approximately $3\frac{1}{7}$. Again, when learning to use protractors, children may form the opinion that the sum of the angles of certain triangles is equal to 180°. Ask, "Will this always be so? Must this always be so? Why?" and when they have been led to appreciate such a simple proof as that by rotation, our pupils have advanced from an opinion about certain triangles to knowledge about all triangles, knowledge which will be of value in designing tiled flooring or in making a hexagonal box. Elementary Geometry gives good practice in this kind of work. The revision in upper standards of reasons for rules formulated in lower standards is also useful. Children who are led to discover afresh the reasons for their decimal or vulgar fraction rules have acquired knowledge which will give them power in tackling problems involving any of these. But here, too, the training must on no account be confined to Arithmetic; children's confident assertions concerning matters of which they know little should be questioned, and so transformed into true knowledge. Arithmetic, however, affords an excellent drilling-ground.

There is another type of thinking which is very common in the solution of Arithmetical problems, and which is of great value in ordering the details of our ordinary lives. A choice is offered to us. It may be a very simple matter: "Shall I buy this brown felt hat or that navy straw one?" Or it may be one which strikes down to the very roots of our life: "Shall I go on teaching, or shall I accept this offer of marriage?" At first we are baffled: the temptation is to make a haphazard choice, and live to think ourselves either extremely lucky or extremely ill-used people! We all know the kind of person who rushes on in life without thought. What is the exact nature of the necessary thinking? I don't know which hat to buy, but I should know if I knew with what clothes I should be wearing it. I shall want to wear it with my navy serge and my brown waterproof, and it must also go with my new winter frock. I could therefore decide if I knew the colour of my new frock. I find that I have already almost unconsciously decided, if possible, to have a green frock; the navy straw would go badly with green, therefore I can now decide to buy the brown felt hat. We all know this type of "backwards reasoning." I should know *a* if I knew *b*, I should know *b* if I knew *c*; but I do know *c*, therefore I know *b* and therefore *a*.

In Arithmetic such reasoning is the veriest commonplace. Set small children the question: "If I can take three children by train to Bradford for ninepence, how much will it cost me to take five children?" The outline of the work is familiar. What must we know to find the cost for five children?—The cost of each ticket.—Can we find the cost of one ticket?—We know the cost of three tickets, therefore of one ticket, and therefore of five tickets. In fact, the general method of solving any Arithmetical problem which is not quite straightforward is to say: "I should know the answer if I knew something else, say *a*: I should know *a* if I knew something else, say *b*. But what is given me tells me *b*, and so *a* and so the answer." Each time that our pupils think out such problems for themselves, instead of having the solution shown to them, they receive practice in a type of thought of the greatest importance in life.

Certain difficulties in the way of securing this valuable mental training for our children will at once suggest themselves. First, children cannot think out everything for themselves: new methods are too hard and must be demonstrated to them. As indicated above, while frequently the complete discovery of the new method is an impossibility, yet if the new work be subdivided into wisely graded steps, each step may be solved independently by at least part of the class, and the whole class may receive some practice in thinking, albeit unsuccessful. Here we may say that the way as a whole is much too long and difficult for our children to traverse alone: we teachers, however, divide it into easy stages, each stage only just out of sight of the one before. Thus guided without being carried, our pupils can on their own legs accomplish the journey and at the same time develop energy and muscle. But the division of the journey into suitable stages is not always easy: it is hoped that Part II. may here give some assistance.

Not all children have the same mental endowment: we know the pupils who can tackle any problem and those who can tackle none. The truth is that nearly every child (certainly 75 to 90 per cent. of our pupils) can learn to work problems independently if we set each child a sufficiency of problems just beyond his immediate grasp. What is good training for the weaker half of our class is poor stuff for our quickest pupils: what affords good training for these quick little people can only be demonstrated to the weaker intellects. This is the old troublesome problem in class management, of which we again postpone the discussion. But the thrill of successfully solving a problem should be a frequent experience of quite three-quarters of every class.

Another very real difficulty is, of course, lack of time. It certainly takes longer to guide children on their own legs along a rough piece of road than it takes to pick them up and carry them. Many teachers wish their children to be taught by sensible, up-to-date methods, but few have the courage to waste time on letting their pupils find out these methods for themselves under wise guidance. There is a twofold reply to this. First, the time wasted early in the course, say in the lowest standards, and with upper classes in the first week of learning any new method, will later be made up without difficulty. This is the experience of those who in faith have made the venture. It is a proof to be appreciated only by those who have a little faith. Make but one or two thoroughgoing experiments in this direction and your faith will grow. Only we must be prepared to waste time in the initial stages and be content to estimate progress over long periods. The other reply is that the mental training is worth a sacrifice in the amount of work covered. If we keep both aims before our eyes, even in the initial stages what is accomplished will not seem small to us, because, besides the results to be read in our children's exercise books, we shall glimpse results written in themselves. It seems almost hopeless to preach to the unconverted of these matters. They are imprisoned within the four high walls of results which show, over-crowded syllabuses, conservatism and sheer hopelessness. Only the trumpet-blasts of those who have the faith will ever cause these walls to fall flat. This is at the heart of educational failure.

Certain results in mental training are, however, achieved by almost every teacher of Arithmetic, results of great value. It is no small matter that our children should receive a daily training in making effort and in concentration. These things are, of course, of little value if obtained in Arithmetic alone, but this subject does make once more an excellent drilling-ground. It is impossible to dawdle through Arithmetic lessons unless they are very badly taught indeed. The difficulty here is that we are often tempted to use lower motives for securing effort and concentration, so that the good accomplished in one direction is in danger of being overbalanced by the harm produced in another: for example, the harm produced by working because of fear. It is worth while to quote a startling paragraph from an article by the school medical inspector to the Liverpool District of the Lancaster County Council. He says, speaking of the psychological characteristics of school children, that little can be said because, individual phenomena being most conspicuous, generalisations are unsafe. "It has, however, often occurred to me that perhaps the emotion which strikes one's attention most frequently and most vividly among school children is Fear. It is a most unpleasing discovery to find out that with a large proportion of school children their attitude to anything which is unusual is one of suspicion and fear, instead of trust and confidence. This is no doubt very largely due to the injudicious application of unsuitable forms of punishment both at home and also at school." That a doctor should be impressed in this way will not surprise many of us who know something of the motives for which children have to work in Arithmetic, but coming, as it does, from an impartial authority, should it not cause us all furiously to think?

It is of the utmost value that in Arithmetic children receive a training in accuracy. The experience of a piece of work being either right or wrong is a real factor in education. Our lessons ought to help to set up in our children's minds a standard of exact knowledge and precise thinking which will permanently affect their mental attitude.

It is strange how futile appears to be any conflict between the two sets of aims indicated above, between those teachers whose banner is utility and those whose banner is training. The two dovetail into each other so perfectly that it is easier to hold the two at once than to hold either separately. Being able to think pays, and also adds to the value of the child to himself and the race. Accuracy is demanded by the business world, and also by those idealists who care that right and wrong should be as distinct as black and white. Again, if we teach only what will pay, we shall discard much that is useless in our syllabuses. Getting rid of all useless material means the abolition of all sums so long and complicated that adults would work them by tables, all long reductions, the H.C.F. rule, cube root and often square root, with all unnecessary weights and measures. Some schools can get rid of more than others. The district and the type of school will affect our syllabuses, and, if possible, all children in the same school will not be forced to cover the same amount of ground. This can all be advocated on grounds of utility. But note the dovetailing again! Every abolition of material because it does not pay means extra time for teaching what is retained so sensibly that the desire to give mental training shall be satisfied. There is no need to teach any material simply because it provides training; retain only the necessary, but teach that in such a way that the child thinks independently and makes discoveries for himself. We can at one and the same time prepare a child to take his place in the society to which he belongs and develop that child's mental endowment to its utmost capacity.

PART II

CHAPTER I

NUMBER IDEAS AND THEIR NOTATION

Introductory Note.

The foundations of this work have usually to be laid by the teachers of very young children. This book makes no pretensions to contributing to the art of teaching little ones. While it aims at giving a general outline which may be of use to teachers of older pupils, it will be sufficient to refer others to some of the books which give either more detailed explanations of the origin and development of number ideas, or more interesting accounts of the development of notation.

e. g. For Origin and Development of Number Ideas, see—

"The Psychology of Number." MacLellan and Dewey. Appleton. 4*s*. 6*d*.

"Study of Mathematical Education." Benchara Branford. Oxford. 4*s*. 6*d*.

For Development of Notation, see—

"Story of Arithmetic." Sarah Cunnington. Sonnenschein. About 3*s*. 6*d*.

"Lectures on the Logic of Arithmetic." M. E. Boole. Oxford. 2*s*.

"History of Elementary Mathematics." Cajori. Macmillan. About 5*s*.

Number Ideas.

In a previous chapter we saw how men were compelled to the quantitative study of the universe by the limits imposed on them; such a quantitative study is called "Arithmetic." In carrying through such a study ideas of numbers are somehow formed in our minds; these ideas can be formed only in studying some aspect of the universe. In fact, we have already emphasised the point that number ideas can be formed only in connection with various activities, including the use of a variety of concrete apparatus. In *action* number ideas have their rise.

But neither the showing of 5 pens nor of 5 yards nor the writing of the symbols 5 or V is the equivalent of grasping the idea of the number 5. The abstract idea of 5 is a possession of the mind. We can have no proof that any child has reached such an idea except the inference which we may draw from his ability to deal independently with a new activity involving the appreciation of the number 5. To teach the beginnings of Arithmetic is thus a most delicate operation.

We have already drawn a contrast between the quantitative study of some concrete reality and the abstract ideas of number formed in the mind in connection with such study. Yet another contrast must be noted, that between these abstract ideas and the figures which are their symbols. Just as we may have a vague idea of darkness and shrink from darkness before we know the word "darkness," so we may have a vague idea of three or five before we know the words "three" or "five," and a less vague idea for some time before we know the symbolic figures 3 and III or 5 and V. But just as knowing the word "darkness" gradually makes our idea of darkness more definite and also permits of communication with other people, so knowing symbols for the number ideas in our minds, tends to make these ideas more definite and also makes communication with others a possibility. The old mechanical plan of teaching arithmetic by starting with figures and teaching us to add, subtract or multiply figures was like teaching a child to read scientific jargon which conveys no real meaning to him. Fortunately to some extent children's experience outside school helped them to form the ideas which gave meaning to the figures; now we aim at giving them, inside school, definite help in forming number ideas before introducing figures as such.

In order to see more clearly the connection between the quantitative study of the universe and the formation of abstract number ideas, let us analyse a few presentations quantitatively. We see before us a heap of bank notes, or a class of boys, or a piece of cloth, or some tea. In each quantitative analysis we have three distinct aspects—

Quantity to be measured.	Quantitative unit by which to measure.	Abstract number telling how often unit is contained in quantity.
Heap of bank notes	a £5 note	4 (£5 notes)
Class of boys	a boy	25 (boys)
Cloth	a yard	3 (yards)
Tea	a 2-lb. packet	8 (2-lb. packets)

Note in passing that the unit need not necessarily involve unity. We can have such derived units as £5 note, 2-lb. packet, as well as such simple units as boy or yard. The important point, however, is that measurement, in order to attain some desired end and the selection of a definite unit by which to measure, are the natural activities out of which abstract number ideas spring.

The Historical Development of Notation.

1. Symbols were used at first to represent, not the abstract number idea, but the concrete quantity to be measured; use was made of fingers, shells, pebbles or counters, each to represent one object. As the sheep left the fold, the shepherd could make a pile of pebbles, one for each sheep, and then as the sheep came home, by removing one pebble for each, he could check the return of his flock. Here a pebble is a symbol for a sheep. Later a new development arose; the pebble might represent one sheep or ten sheep according to its position. Perhaps, as Mrs. Boole suggests, the shepherd's little daughter came to help. As each sheep passed him, the shepherd raised a finger, for this could be done more quickly than lifting a pebble; when ten sheep had passed, the father could go no farther, but his daughter could raise one of her fingers to show that one set of ten sheep had passed. He counted the next ten; she raised a second finger. Thus 3 of her fingers and 2 of his would be a symbolism for 32 sheep. If the flock was

greater than a hundred sheep, doubtless the shepherd's son might allow one of his fingers to represent a hundred sheep! Whether this be historically correct or merely an interesting suggestion, it is certainly true that in time men began to use their counters economically and in such a way that calculation became easier. Various forms of the abacus were used by Egyptians, Hindus, Greeks and Chinese as they are still used in Russia, China, and our own infant schools. In the abacus the counters are arranged in columns. Two counters in one column or 2 beads on one wire mean 2 units; in the next column, 2 tens; in the next, 2 hundreds. This makes calculation possible without written symbols, as the values of the counters can be seen at a glance and the number of counters in one column can never exceed nine. This, being the principle of the ball-frame and of other educational abaci, need not be elaborated. It is interesting to note that this primitive method of recording numbers has left a permanent mark on our language, for "calculation" is derived from "calculus," the Latin word for a pebble.

2. In part simultaneously with this stage, arose another, in which men invented written symbols to represent the abstract number ideas. At first, however, these written symbols were used only for recording numbers; they gave no help in calculation, which had still to be done with the abacus. Let us look at a few of them, to understand the force of this last sentence.

The Roman symbolism is familiar, founded as it is on I, V, X, L, C, D, M. The Greeks used the twenty-four letters of their alphabet along with three Hebrew letters in order to have twenty-seven symbols, used to represent units to 9, tens to 90, and hundreds to 900. For instance $\alpha = 1$, $\beta = 2$, $\gamma = 3$; $\iota = 10$, $\varkappa = 20$, $\lambda = 30$; $\varrho = 100$, $\sigma = 200$, $\tau = 300$. So that $\alpha\tau$ or $\tau\alpha = 301$; $\sigma\lambda$ or $\lambda\sigma = 230$. Note the difficulty of working, say, addition sums with these notations—

XLVII	$\sigma\alpha\varkappa$
CCLXXIX	$\gamma\iota$
MCDIV	$\tau\beta$
———	——

The Greeks and Romans would naturally continue to perform their calculations with pebbles on the abacus, finding, however, these symbols most useful for recording numbers. While these and other written symbols for number ideas were developing, those which finally became our figures 1 to 9 were taking their rise in India; they appear as early as 150 B.C., and some of them, at any rate, seem to have been the first letters of the words corresponding to our "four," "five," or "seven." Naturally in manuscript their forms varied, so that their origin tends to be shrouded in mystery; certainly they did not acquire their present fixed form until the invention of printing made permanent much that was formerly in flux.

3. Finally, however, this written notation became valuable for calculations by developing into a notation based on place value, with ten as the most common basis for grouping; ten probably owes its prominent position to the number of our fingers or toes. Our present place-value notation is, in fact, a written picture with the symbols 1 to 9, of the appearance of the abacus, with its counters arranged in columns. Such a picture as |7| |3| |2| would at once call up the appearance of the abacus with its 7 counters, 3 and 2 counters in the three columns; similar pictures written below in corresponding columns would lead at once to calculations being done, not with the concrete symbols for the real objects, but with the written symbols for the number ideas. The gain in speed would encourage the development of this written mental calculation as a good alternative to the use of the abacus.

At first there was a serious drawback to speed and accuracy. Sometimes in performing a calculation on the abacus, the tens column, say, would have no counters left in it, the column itself remaining, of course, clear and distinct. In writing, however, especially when the notation had become contracted to 732, this was a serious difficulty; the correct 8 5 became quickly the incorrect 85. Then probably a dot was used to signify the empty column. But the dot was too trivial in manuscript to serve the purpose and so was enclosed in a ring to prevent its disappearance. At last, however, the dot did disappear; but the ring was left, a symbol for nothing, a symbol invented only when written calculation had revealed the necessity for something of the kind. Its origin is thus different from that of the symbols **1** to **9**, but its appearance completed the amazingly simple and powerful Arabic notation, which is now used by the great majority of the peoples of the world.

Applications to Teaching of Notation.

It is obvious how closely connected with the experience of the race in learning notation is the experience of children. But the final result reached is so beautifully simple and useful that it is wise economy of time to condense the experience of the race, in order swiftly to reach this. With this reservation, let us note certain points in the teaching of notation, emphasised by the historical outline.

1. For small children, tens and hundreds should have concrete representation. Such a great variety of concrete aids is in use in our schools that it is necessary to note the fundamental differences among them; these differences correspond to three distinct stages of development.

(*a*) We have apparatus where a ten consists of ten ones, or a hundred of ten such tens: *e. g.* bundles of sticks or tram-tickets, bunches of daisies, beans in a bag, beads on a thread, a strip formed of **10** squares and large squares formed of **10** such strips. The laying of shells or counters in groups does not emphasise sufficiently the ten as a new unit, though such apparatus may be very useful in dealing with smaller numbers.

(*b*) We have apparatus where a ten, because of some obvious quality, usually size or colour, is different from a one and may be *assumed* to be equal to ten ones: *e. g.* counters of different colours, beads of different colours or sizes, shells of different sizes. The use of such apparatus can easily be extended to represent hundreds and thousands as well as tens. The equivalence of **12** pence and **1** shilling is a parallel case.

(*c*) We have apparatus where hundreds, tens and units take their value only from their place or position: *e.g.* any form of abacus, including the common ball frame. Place value can, however, be introduced in connection with (*a*) or (*b*) if children become accustomed to the arrangement of their hundreds, tens and units in an orderly sequence.

2. While at first calculations should be performed concretely, we must aim at quickly substituting for this, written and mental work, depending on the use of the symbolic figures. The advance should be gradual, but real, so that in the end no apparatus is necessary. One great objection to finger-counting is that fingers cannot be removed. The history of the race shows clearly that the retention of concrete aids after a certain period means a distinct loss in speed and accuracy. The differences in individual children are, however, so marked that it is impossible to fix an age or standard beyond which no apparatus should be necessary.

3. In teaching written notation we should remember its gradual development in history. As children begin to form number ideas, they can with advantage be taught the symbolic figures. There is no harm, but great gain in the use of figures if they have a meaning to the child. Zero, however, is a symbol with which children always have difficulty. Is not the cause that we try to teach it as we teach the other symbols, whereas historically it arose only to meet a felt need in calculation? If the children at first put their figures in "nests" or "boxes" to hold hundreds, tens and units, and if some day there is no figure for the units nest, and the tens nest looks as if it were a units nest, they will see the need for retaining the empty nest. In time they will know that such an "empty nest" is always put down when there is nothing at all, and they will have some vague notion of the reason for using **0**. Another plan is to allow children to leave a blank until a mistake is made; probably of themselves they will insert a dot, and can then easily be led to encircle the dot with a ring and so reach finally the common use of zero. Older

children will be interested in the story of the development of zero in some such way as—

$$\begin{array}{cccc} \boxed{4} & & \boxed{3} & \boxed{5} \\ 4 & & 3 & 5 \\ 4 & \cdot & 3 & 5 \\ 4 & \odot & 3 & 5 \\ 4 & 0 & 3 & 5 \end{array}$$

4. If the historical outline has succeeded, it will have emphasised the wonderful beauty and simplicity of the Arabic notation which we use. It is part of a child's education to realise this, but the full realisation is possible, naturally, only for older children. It is good to revise notation before beginning the thorough treatment of decimals common in Standard V. Our pupils have to learn how to read Roman notation; let them try to work simple sums with it. They can then discover why it is so much harder for calculation than our Arabic notation, and the resulting understanding of the principles underlying our common notation will be the best possible preparation for sensible work with decimals.

CHAPTER II

THE FUNDAMENTAL OPERATIONS—ADDITION

Few difficult or doubtful questions arise in connection with the teaching of addition, but some miscellaneous notes may be given on various points.

Meaning of Addition.

Addition is the mental operation performed in the massing together of concrete quantities of the same kind. The mental idea can be taught only through a variety of work with concrete apparatus. The concrete operation must be performed with some definite end in view: *e. g.* keeping scores in easy games, finding how much of some treasure a child possesses, playing shop. The sound laying of this foundation will avoid trouble over the addition of quantities measured in different units; hundreds will go with hundreds, tens with tens, units with units, without difficulty. In time the concrete can be imagined instead of handled; still later, drill with abstract numbers is safe, if pupils realise the purpose of the drill to be the development of speed and accuracy.

Addition Tables.

In upper standards addition is frequently weaker than multiplication. The cause for this is that, while all understand that skill in multiplication is based on a sound knowledge of multiplication tables, not all realise that skill in addition is based on a thorough acquaintance with a limited number of simple combinations.

1 + 1
2 + 1, 2 + 2
3 + 1, 3 + 2, 3 + 3
4 + 1, 4 + 2, 4 + 3, 4 + 4

---

9 + 1, 9 + 2, 9 + 3, 9 + 4, 9 + 5, 9 + 6, 9 + 7, 9 + 8, 9 + 9

It is frequently said that all addition should be done through the ten; inspectors have been known to find fault with children who give $17 + 8$ immediately as 25, instead of $17 + 3 = 20$; $20 + 5 = 25$. All addition must be taught at first through the ten; children must learn that in adding 7 sticks and 5 sticks they put 3 of the 5 with the 7, to make a ten-bundle, and have 2 single sticks left: *i.e.* $7 + 5 = 7 + 3 + 2 = 10 + 2 = 12$. But they cannot be said to know addition until they have repeated this so often that $7 + 5$ immediately suggests 12; just as 5×7, though at first thought of as $7 + 7 + 7 + 7 + 7$, comes automatically to suggest 35. Not to master these simple combinations means that throughout Arithmetic two operations are performed instead of one. The intermediate steps should gradually disappear.

On the mastery of these combinations depends all skill in addition. For children can see that Mary's 2 sticks and John's 5 sticks put together give as many sticks as Mary's 5 and John's 2; *i.e.* $2 + 5 = 5 + 2$. Again, take such examples as $27 + 8$: 7 and 8 units at once suggest 15 units; the one ten extra comes to be automatic: *i.e.* 27 and 8 suggest 5, 35. This is shorter than 27, 30, 35. Addition of any number of columns depends ultimately on this ability to add to a number of not more than two figures a number of one figure.

The only way to combat ignorance of an addition combination is to fall back on a ball-frame or some other piece of apparatus.

Drill in Addition.

Here the great essentials are variety and a recognition of the fact that children see sense in becoming perfect in addition. Common artifices for class work may be noted.

1. Teacher writes circle of figures on the blackboard, touches two or more of these at random; children reply either orally or by writing down answer.

2. Teacher can name a string of two or more numbers, children writing down answer.

3. Children practise rapid addition (and subtraction) by

equal increments : *e.g.* if teacher starts 5, 8, 11, children should continue 14, 17, 20, 23, . . . up to 100; or if teacher begins 93, 91, 84, children should go on 77, 70, 63, . . . down to 0. Taking equal increments alternately is better fun : *e.g.* add 3 and 5 alternately,—6, 9, 14, 17, 22, 25. . .

4. Matches between two halves of a class or between smaller groups working under captains, go far to increase interest.

5. Tots are, of course, indispensable.

6. Children are taught to check addition by adding the numbers in the opposite direction. They should also practise adding in rows as well as in columns.

Carrying Figures.

If the carrying figure in addition must be set down, it should be remembered that the worst possible place for it is at the side of the sum, where its meaning becomes obscure, and where it is necessary sometimes to work the whole sum in order to correct a mistake in one column. A good plan is to set it down as a small figure at either the top or bottom of its own column in the sum.

$$
\begin{array}{l r}
\textit{e.g.} & \text{soldiers} \\
 & 4\;5\;3 \\
 & 7\;8\;9 \\
 & 6\;2 \\
 & 3\;9\;1 \\
 & {\scriptstyle 2}\;\;{\scriptstyle 1}\;\;\; \\
\hline
 & 1\;6\;9\;5 \text{ soldiers.}
\end{array}
$$

Two-Column Addition.

Skill in mental addition should involve the ability to add quickly any pair of numbers of two figures, such as 75 and 48. Suitable steps would be—

(*a*) Answer less than 100—

1. Adding tens only : *e.g.* $75 + 20$; $48 + 40$.
2. Adding tens and units: *e.g.* $75 + 23 = 95 + 3 = 98$.

(*b*) Answer greater than **100**—

1. Adding tens only: *e. g.* $75 + 60$; $48 + 90$.
2. Adding tens and units: *e.g.* $75 + 69 = 135 + 9 = 144$.

N.B. By thus subdividing the work into a series of graded steps children can learn the quickest method naturally. After the hardest combinations have been mastered, they can analyse and formulate the method; *i.e.* add the tens to the first number and to that sum add the units.

Older children keen on Arithmetic find practice in two-column addition an entertaining exercise: *e. g.*—

2 4 6 3	$59 + 72 = 131$
1 4 8 9	$131 + 89 = 211 + 9 = 220$
3 6 7 2	$220 + 63 = 283$
1 3 5 9	
8 9 8 3	Carry 2. $15 + 36 = 51$
	$51 + 14 = 65$
	$65 + 24 = 89$

CHAPTER III

THE FUNDAMENTAL OPERATIONS—SUBTRACTION

For the successful teaching of subtraction the only things needful are a complete mental grasp of the method chosen and an understanding of its peculiar difficulties. By explaining the great variety of possible methods and estimating the value of each, this chapter hopes to help some teachers to decide between rival claimants and to encourage others to experiment on a fresh method if one commends itself as possibly more useful than the old friend.

Let us divide the ground to be covered into three stages—

1. Subtraction of numbers each less than **10**.
2. Subtraction of numbers, one less than **10**, the other between **10** and **20**.
3. Subtraction of larger numbers.

We must remember that the results of discussing the first two stages are involved in the methods of the third stage. Throughout, for brevity, we shall usually omit reference to the concrete, taking for granted that it must be used in the initial attacks on any method.

1. *Numbers each less than* 10.

This, the simplest stage of all, involves us in the distinction between the method of "direct subtraction" and that of "inverse addition."

(*a*) Direct Subtraction : *e. g.* Take **5** from **8**. Symbolically $8 - 5 = ?$

(*b*) Inverse Addition : *e. g.* What must be put to **5**, to make **8**? Symbolically $5 + ? = 8$.

Note that these are based on distinctly different activities. In (*a*) 8 sticks are laid down: 5 are then set aside: 3 are left. In (*b*) 5 sticks are laid down; and then 3 extra ones are placed beside them, to give finally 8.

These methods are, however, not antagonistic in their claims. Each pupil must realise the two activities which lead to the same mental operation termed "subtraction." But (*b*) has the advantage of being so closely connected with addition that it can be taught along with it. Probably then the first emphasis should be on the "inverse addition" idea, the mental operation being evoked in connection with this activity; a little later the child will discover that the same mental operation is evoked by the "direct subtraction" activity.

2. *One number less than* 10, *the other between* 10 *and* 20.

This stage, while including (1), involves us in a further variety of method. Shall we teach our children to work through the ten, or to work without explicit reference to the ten?

(*a*) THROUGH TEN: *e.g.* $13 - 8 = ?$ is worked as $10 - 8 = ?$, $? = 2$, $2 + 3 = 5$;
and $8 + ? = 13$ is worked as $8 + ? = 10$, $? = 2$, $2 + 3 = 5$.

(*b*) WITHOUT TEN: *e.g.* $13 - 8 = ?$, $? = 5$;
and $8 + ? = 13$, $? = 5$.

Here, too, we have no real antagonism between the methods, ideal teaching finding room for both. Every child must at first work through the ten, as not otherwise can notation be understood. But when addition combinations have been mastered, he should become able to deal with subtraction without direct reference to the ten, the ultimate gain in speed being obvious.

3. *Larger Numbers, involving written Notation.*

This final stage, leading in written work to the difficulty of taking away a number from a smaller number just above it, brings to our notice three fundamentally different methods, one of these again branching out into two. We may summarise the three as—

(*a*) Two Methods of Decomposition.
(*b*) Method of Equal Additions.
(*c*) Method of Complementary Addition (or Inverse Addition).

(*a*) and (*b*) can be based on any of the methods of (1) and (2), but (*c*) is by its nature committed to the inverse addition plan for small numbers.

(*a*) Methods of Decomposition

(i) *Decomposition before Subtraction.*

$$e.g.\ \left.\begin{array}{cc} \text{T.} & \text{U.} \\ 6 & 5 \\ 2 & 9 \end{array}\right\} \text{ is equivalent to } \left\{\begin{array}{cc} \text{T.} & \text{U.} \\ 5 & 15 \\ 2 & 9 \\ \hline 3 & 6 \end{array}\right.$$

$$\left.\begin{array}{ccc} \text{H.} & \text{T.} & \text{U.} \\ 7 & 3 & 1 \\ 2 & 7 & 5 \end{array}\right\} \text{ is equivalent to } \left\{\begin{array}{ccc} \text{H.} & \text{T.} & \text{U.} \\ 7 & 2 & 11 \\ 2 & 7 & 5 \end{array}\right\} \therefore \text{ is equivalent to } \left\{\begin{array}{ccc} \text{H.} & \text{T.} & \text{U.} \\ 6 & 12 & 11 \\ 2 & 7 & 5 \\ \hline 4 & 5 & 6 \end{array}\right\}$$

$$\left.\begin{array}{cccc} \text{Th.} & \text{H.} & \text{T.} & \text{U.} \\ 6 & 0 & 0 & 0 \\ 3 & 1 & 7 & 2 \end{array}\right\} = \left\{\begin{array}{cccc} \text{Th.} & \text{H.} & \text{T.} & \text{U.} \\ 5 & 10 & 0 & 0 \\ 3 & 1 & 7 & 2 \end{array}\right\} = \left\{\begin{array}{cccc} \text{Th.} & \text{H.} & \text{T.} & \text{U.} \\ 5 & 9 & 10 & 0 \\ 3 & 1 & 7 & 2 \end{array}\right\} = \left\{\begin{array}{cccc} \text{Th.} & \text{H.} & \text{T.} & \text{U.} \\ 5 & 9 & 9 & 10 \\ 3 & 1 & 7 & 2 \\ \hline 2 & 8 & 2 & 8 \end{array}\right.$$

This method is obviously one which children can easily discover. If presented with 6 ten-bundles and 5 single sticks and asked for 29, children very likely will untie or "decompose" a ten-bundle to obtain the 9 single sticks. This simplicity of idea is a great advantage. In general there is a corresponding simplicity of manipulation. The rule is the simple one of diminishing the figures in upper line by 1 if in previous column the lower figure is greater than the upper one. But when there is a row of 0's in the upper line, the decomposition of the higher units becomes more complex, and the rule for manipulation is so complicated with its 9's and 10 at the end, that it is apt to lead to errors.

In short, in spite of its general simplicity of idea and manipulation, in practice we obtain disappointing results in speed and accuracy from older children taught by this method. Recent experiments carried out by P. B. Ballard in London schools and reported on in the *Journals of Experimental Pedagogy* for December 1914 and March 1915, bring out forcibly the marked superiority in results of the method of equal additions over the method of decomposition.

(ii) *Decomposition after Subtraction.*

The general idea underlying this less familiar form of decomposition may best be brought out by considering again the child with 65 sticks asked to give 29. The child would be just as likely first of all to take the two ten-bundles from the 6 ten-bundles, leaving 4 ten-bundles and the 5 single sticks. Then, finding himself unable to take 9 from 5 sticks, he would break up one of these remaining 4 ten-bundles, leaving 3 ten-bundles and ultimately 6 single sticks.

$$e.g.\ \left.\begin{array}{cc} \text{T.} & \text{U.} \\ 6 & 5 \\ 2 & 9 \\ \hline 4 & \end{array}\right\} = \left\{\begin{array}{cc} \text{T.} & \text{U.} \\ 4 & 5 \\ & 9 \\ \hline & \end{array}\right\} = \left\{\begin{array}{cc} \text{T.} & \text{U.} \\ 3 & 15 \\ & 9 \\ \hline 3 & 6 \end{array}\right. \quad \text{starting from left-hand side}$$

$$\left.\begin{array}{ccc} \text{H.} & \text{T.} & \text{U.} \\ 7 & 3 & 1 \\ 2 & 6 & 5 \\ \hline 5 & & \end{array}\right\} = \left\{\begin{array}{ccc} \text{H.} & \text{T.} & \text{U.} \\ 7 & 13 & 1 \\ 2 & 6 & 5 \\ \hline 4 & 7 & \end{array}\right\} = \left\{\begin{array}{ccc} \text{H.} & \text{T.} & \text{U.} \\ 7 & 13 & 11 \\ 2 & 6 & 5 \\ \hline 4 & 6 & 6 \end{array}\right. \quad \begin{array}{c}\text{starting with} \\ \text{hundreds.}\end{array}$$

At this stage it may be well to explain the rule. Start at left-hand side and always look ahead one place. If no difficulty appears in next place, write down number as obtained; if difficulty appears ahead, write down one less than number obtained by subtraction—

e.g. 7 3 1 (1) (7—2)H = 5H : write down 4H, because 6
2 6 5 is greater than 3 in tens column.

4 6 6 (2) (13—6)T = 7T : write down 6T, because 5 is greater than 1 in units column.

(3) (11—5)U = 6U : write down 6U, because no difficulty follows.

Note that this leads to no difficulty with a row of 0's.

e.g. 6 0 0 0	(1) 6—1 = 5,	write down	4Th.
1 2 3 7	(2) 10—2 = 8,	„ „	7H.
	(3) 10—3 = 7,	„ „	6T.
4 7 6 3	(4) 10—7 = 3,	„ „	3U

That this overcoming of the old difficulty is more apparent than real can be seen from the following example.

6 1 3 5 (1) 6—2 = 4 : write down 4Th, since next column
2 1 3 8 is easy.

(2) 1—1 = 0 : write down 0H, since next column
4 0 0 is easy.

(3) 3—3 = 0 : what is to be written down, since 8 is greater than 5 in units column ?

The remaining 4000 has to be decomposed into 3Th, 9H, 9T, 10U as before. This difficulty need not be elaborated.

The supporters of this method claim that it saves time to write figures from left to right and that it emphasises the most important part of the result. In other respects its position is obviously similar to that of "decomposition before subtraction."

(*b*) METHOD OF EQUAL ADDITIONS

The fundamental idea underlying this method can best be illustrated by a simple example.

A is 7 years old, B is 5 years old : A is 2 years older than B, *i. e.* 7 — 5 = 2.

3 years hence. A is 10 years old, B is 8 years old : A is 2 years older than B, *i. e.* (7 + 3) — (5 + 3) = 2.

10 years hence. A is 17 years old, B is 15 years old : A is 2 years older than B, *i. e.* (7 + 10) — (5 + 10) = 2.

In short, if we increase each of two quantities by the same amount, the difference between them remains the same. Many illustrations can be found. Let us apply this to graded examples involving the subtraction difficulty.

$$\left.\begin{array}{cc}\text{T.} & \text{U.}\\ 6 & 5\\ 2 & 9\end{array}\right\} = \left\{\begin{array}{cc}\text{T.} & \text{U.}\\ 6 & 15\\ 3 & 9\\ \hline 3 & 6\end{array}\right.$$

adding **10** units to one quantity and an equivalent one ten to the other.

$$\left.\begin{array}{ccc}\text{H.} & \text{T.} & \text{U.}\\ 7 & 3 & 4\\ 5 & 6 & 8\end{array}\right\} = \left\{\begin{array}{ccc}\text{H.} & \text{T.} & \text{U.}\\ 7 & 13 & 14\\ 6 & 7 & 8\\ \hline 1 & 6 & 6\end{array}\right.$$

adding **10** units and **10** tens to one quantity, and an equivalent one ten and one hundred to the other.

$$\left.\begin{array}{cccc}6 & 0 & 0 & 0\\ 1 & 2 & 6 & 4\end{array}\right\} = \left\{\begin{array}{cccc}6 & 10 & 10 & 10\\ 2 & 3 & 7 & 4\\ \hline 4 & 7 & 3 & 6\end{array}\right.$$

Note, first, that we here replace the given question by a *different* one; second, that the result of doing this is to abolish difficulty with any possible combination of figures. We add **1** to the lower line, if previous column has had its lower figure greater than its upper figure, and this rule is absolutely without exception.

This simplicity of manipulation is the great advantage of this method: it gives excellent results in speed and accuracy. Its one serious drawback is that the fundamental idea is not a simple one likely to be obvious to children, but one requiring careful preliminary treatment. We cannot too strongly condemn the common practice of covering up this difficulty by the use of the terms "borrow" and "carry," which are supposed to afford an explanation because they are familiar words. If one has to take away **29** sticks from **65**, in general one has no "bank" from which to borrow **10** sticks. Even if we imagine ourselves to possess such a "bank," we should have to talk of "paying back" the extra ten sticks borrowed, not of "carrying" a ten. "Adding" ten is in each case the simplest expression provided that the underlying idea has, to some extent, been realised.

(c) Method of Complementary Addition, or Inverse Addition

This method involves carrying out completely the idea involved in $5 + ? = 8$; $? = 3$. It is a method little used in this country, but where it has been tried, the results give reason for further experiment. A simple example will explain.

Mary has **27** sticks. How many extra sticks must she receive, to have **69**? Obviously she will require **2** sticks and **4** ten-bundles. This can at first be set down as an addition sum with a missing line—

$$\begin{array}{r} 2\ 7 \\ .\ . \\ \hline 6\ 9 \end{array} \quad \text{leading to} \quad \begin{array}{r} 69 \\ - \\ 27 \\ \hline .. \end{array} \quad \text{or} \quad \begin{array}{r} 69 \\ 27 \\ \hline .. \end{array}$$

If we alter Mary's final total to **64** sticks, the question becomes more difficult. Certainly we cannot add any number of sticks to **7** to get **4** sticks. All we can do is to add **3** sticks to make with the given **7** one of the ten-bundles required, and **4** more to give the **4** single sticks required; in other words, as we cannot make up to **4** we make up to **14**.

We now have **3** ten-bundles and **4** sticks; add **3** more tens to give **64** sticks—

$$i.e.\ \begin{array}{r} 6\ 4 \\ \hline 2\ 7 \\ .\ . \end{array}$$

(1) *Units.* $7 + ? = 4$—impossible; $\therefore 7 + ? = 14$; $? = 7$.

(2) *Tens.* We now have the **4** sticks and also an extra ten towards the **6** tens required. $(1 + 2)T = 3T$, $3 + ? = 6$, $? = 3$.

Children can discover the result if given the question to work with apparatus or as an addition sum with a missing line.

Let us take the working of a longer example to make the resulting manipulation clearer.

6 1 8 5 (1) *Units.* $8 + ? = 15$; $? = 7$.
2 6 3 8 (2) *Tens.* Carry 1 ten from 15 units obtained, towards getting 8 tens; $1 + 3 = 4$;
———
. . . . $4 + ? = 8$; $? = 4$.

(3) *Hundreds.* No hundreds to carry from 8 tens obtained; $\therefore 6 + ? = 11$; $? = 5$.

(4) *Thousands.* Carry 1 thousand from 11 hundreds obtained, towards getting 6 thousands. $2 + 1 = 3$; $3 + ? = 6$; $? = 3$.

Result is 3Th, 5H, 4T, 7U, or 3547.

It will be noticed that finally the manipulation is practically the same as for method of equal additions : "carry" one to lower line if in previous column lower figure has been greater than upper figure. Note that "carry" is here used correctly.

This method thus leads ultimately to a very simple rule for manipulation, applicable to every combination of figures, while the idea of inverse *addition* which underlies it is easy to comprehend. That it is, humanly speaking, a natural method to "make up" to the higher number, is shown by the shopkeepers' method of counting change by "making up" from the amount required to the amount tendered. But from the point of view of subsequent work, this method has even greater advantages.

For one thing it makes it possible to take a sum of numbers from one number in a single step, a piece of work useful in connection with money dealings and also in work involving logarithmic tables—

e.g. (i) I have £1635 in bank. I have drawn £493, £176, £230. How much is left ?

£1 6 3 5 (1) *Units.* $0 + 6 + 3 = 9$; make up to 15 by adding 6.
———
4 9 3 (2) *Tens.* Carry 1 from 15 units; $1 + 3 + 7$
1 7 6 $+ 9 = 20$; make up to 23 by adding 3.
2 3 0 (3) *Hundreds.* Carry 2 from 23 tens; $2 + 2 + 1$
. . . $+ 4 = 9$; make up to 16 by adding 7.

$\therefore$ Remainder = £736.

(ii) Find value of 2·3417 − $\bar{3}$·2613 − $\bar{1}$·4179 − 2·3338.

2·3417
$\bar{3}$·2613
$\bar{1}$·4179
2·3338
. . . . .

(1) 8 + 9 + 3 = 20 : make up to 27 by adding 7.
(2) Carry 2 : 2 + 3 + 7 + 1 = 13 ; make up to 21 by adding 8.
(3) Carry 2 : 2 + 3 + 1 + 6 = 12 : make up to 14 by adding 2.
(4) Carry 1 : 1 + 3 + 4 + 2 = 10 : make up to 13 by adding 3.
(5) Carry 1 : 1 + 2 + $\bar{1}$ + $\bar{3}$ = $\bar{1}$: make up to + 2 by adding 3.

Result is 3·3287.

Again, the Italian method of Long Division in which one sets down only the result of subtraction, omitting the multiplication line, is made easy if "making up" the product to dividend is the method chosen—

e. g. 6256 ÷ 17.

```
     3 6 8
17)6 2 5 6
   1 1 5
     1 3 6
       . .
```

(1) 62H ÷ 17 = 3H
3 × 7 = 21 : 21 + ? = 22 : ? = 1
Carry 2 from 22 : 3 × 1 + 2 = 5 : 5 + ? = 6 : ? = 1.
Set down 11 and take down 5 from dividend.

(2) 115T ÷ 17 = 6T
6 × 7 = 42 : 42 + ? = 45 : ? = 3
Carry 4 from 45 : 6 × 1 + 4 = 10 : 10 + ? = 11 : ? = 1
Set down 13 and take down 6 from dividend.

(3) 136U ÷ 17 = 8U
8 × 7 = 56 : 56 + ? = 56 : ? = 0
Carry 5 from 56 : 8 × 1 + 5 = 13 : 13 + ? = 13 : ? = 0, *i.e.* no remainder.

Result is 3H, 6T, 8U, or 368 and nothing over.

From theoretical considerations, then, one would have no hesitation in saying that this Inverse Addition method was the best; but practical results are not likely to verify this conclusion until more schools give it a fair chance by experiment.

CHAPTER IV

THE FUNDAMENTAL OPERATIONS—MULTIPLICATION

BEFORE some analysis of the steps involved in the successful teaching of multiplication, we must touch on two questions of theoretical interest which have a real bearing on practical teaching; we refer to the nature of the operation termed "multiplication" and to the interchange of factors.

The Nature of Multiplication.

What is multiplication? The simplest answer is that we have multiplication when we add the same quantity a certain number of times. Understanding that we have been given 4 sticks 3 *times* implies a further stage of development than that of simply putting together 4 sticks and 4 sticks and 4 sticks as we are given them. Multiplication, then, is fundamentally continued addition, when the new idea of "times" has developed. It connects itself most closely with addition, but is more difficult because of this more abstract idea.

Probably this conception of multiplication is best at first for our children, but unless we have ourselves advanced beyond it, we shall find trouble when the result of multiplying a quantity by a fraction is a diminished quantity; children cannot understand how multiplication can produce a smaller answer. Here we have definitely to extend the meaning we have given to the term multiplication.

In the first steps of teaching multiplication by fractions, the difficulty need not arise for the child. We do not mention multiplication at all. We simply tackle, with the aid of diagrams, such work as $\frac{1}{3}$ of $\frac{1}{5}$ yard, $\frac{2}{3}$ of $\frac{1}{5}$ yard, $\frac{2}{3}$ of $\frac{4}{5}$ yard: see pages 127 and 128 for suggestions as to method of doing this. "Of" makes sense to the child and at first he need not be further troubled.

But sooner or later he must see for himself that multiplication is a good term to apply to this operation, since it dovetails into his previous work in multiplying by a whole number, while giving an extended meaning to the word multiplication. For instance, these steps might be taken—

(1) This line is 8 ins. long. What is the length of $\frac{2}{3}$ *of* this line?

$\frac{2}{3}$ of 8 ins. $= \frac{16}{3}$ ins. $= 5\frac{1}{3}$ ins., as above.

(2) What is the length of three *of* these lines?
3 of the 8 in. lines $= 3 \times 8$ ins. $= 24$ ins.

(3) What is the length of $3\frac{2}{3}$ *of* these lines?
$3\frac{2}{3}$ of the 8 in. lines = 3 of the lines and $\frac{2}{3}$ of a line
$= 3 \times 8$ ins. $+ \frac{2}{3}$ of 8 ins.
$= 24$ ins. $+ \frac{16}{3}$ ins.
$= 29\frac{1}{3}$ ins.

(4) What is the length of a line $3\frac{2}{3}$ *times* 8 ins.?
$3\frac{2}{3} \times 8$ ins. = 3 times 8 ins. and $\frac{2}{3}$ time 8 ins.
$= 3 \times 8$ ins. $+ \frac{2}{3} \times 8$ ins.
$= 24$ ins. $+$?

But the unknown answer clearly ought to be the same as in (3), that is equal to $\frac{2}{3}$ *of* 8 ins. So we can dovetail this new idea of multiplication into our primary idea of multiplication as continued addition. We do this by assuming this new meaning for the term multiplication: multiplying a quantity by, say, $\frac{2}{3}$, is to be regarded as taking $\frac{2}{3}$ *of* that quantity. Thus we can bring the operations demanded by 3 times 8 ins., $\frac{2}{3}$ of 8 ins., $3\frac{2}{3}$ times 8 ins. and $3\frac{2}{3}$ of 8 ins., all under the one term multiplication, by giving this term an extended meaning.

MacLellan and Dewey, who, in their *Psychology of Number*, have done so much to emphasise the natural development of number and the numerical operations, suggest what probably is the primitive measuring need out of which the operation of multiplication sprang, and show that this need would be the same, whether the multiplier be a whole number or a fraction. "Multiplication makes the expression of a measured quantity more complete, by finding the number of primary units in it, given the number of derived units." Take first a simple illustration. A man finds himself with 7 £5 notes, the £5 note being a unit derived from the primary unit, £1. He may have a vague idea of what he can achieve by the use of these 7 £5 notes, although ignorant of their relation to £1. He may feel them more important than 7 £1 notes because of their greater size and stiffness, but exactly how much more important are they? How much more valuable are they as means to a desired end? He certainly makes the expression of the measured quantity of notes more complete when he realises that 7 £5 notes are the same as 35 £1 notes. Or a little child may have a choice between 8 threads of beads each containing 7 beads, and 9 threads each containing 6 beads. Until he has reduced 8 7-bead threads and 9 6-bead threads to 56 and 54 single beads, he cannot estimate their relative importance to him. In this case the derived units are the 7-bead thread and the 6-bead thread; the primary unit is the single bead. Or a girl wishes to buy 4 yards of cloth at 3*s*. 9*d*. a yard, and knows she will have to pay 4 times 3*s*. 9*d*. She will not know whether she can afford it until she has expressed the 4 derived units (3*s*. 9*d*.) in the primary units 1 shilling and 1 penny. The notion of 15*s*. clears her vision at once.

The same is true with fractions. One child has $\frac{2}{3}$ of $\frac{3}{4}$ of a loaf; another has $\frac{2}{5}$ of $\frac{5}{6}$ loaf. We cannot tell which child has most bread until we find out what part of 1 loaf, the primary unit, each has. $\frac{1}{3}$ of $\frac{3}{4}$ loaf $= \frac{1}{4}$ loaf. So $\frac{2}{3}$ of $\frac{3}{4}$ loaf $= \frac{2}{4}$ loaf $= \frac{1}{2}$ loaf. And $\frac{1}{5}$ of $\frac{5}{6}$ loaf $= \frac{1}{6}$ loaf. So $\frac{2}{5}$ of $\frac{5}{6}$ loaf $= \frac{2}{6}$ loaf $= \frac{1}{3}$ loaf. It is now easy to appreciate the greater wealth of the first child.

This attitude to the operation in both its primary and extended meanings, is very suggestive to the teacher. In each development of multiplication we can introduce the work through problems involving the more complete measurement of a quantity for some end; unconsciously the child will catch the right attitude. The place of drill work in multiplication is, of course, enormous, but our plea is that in the initial stages the mental activity of multiplication should be evoked by problems embodying this point of view.

Interchange of Factors.

At some stage most people discover that the result of multiplying 46 by 39 is the same as the result of multiplying

39 by 46. We become skilful in choosing the easier multiplier; or we check our product by using first one factor and then the other as multiplier. What we urge here is, that since the fact that $46 \times 39 = 39 \times 46$ is implicit in the idea of multiplication, use be made of it from the earliest stages, to lighten the labour of learning tables, to check results of multiplication, to solve the mystery of division and to explain the usual methods of setting down reduction sums. We urge that since it is of fundamental importance, its discovery should not be left to chance and time. The teaching of this principle should have an important place in the early stages of number work.

The truth of the principle is obvious if we treat multiplication as the more complete expression of a quantity measured in derived units. If we wish to lay out 5 6-bead threads, we may do so in two distinct ways. We may lay out 6 beads, once, a second time, a third time, a fourth time, a fifth time; we 5 times lay out 6 beads. But we may also lay out 5 beads one to each thread, then another 5 beads, a second to each thread, another 5, a third to each thread, until we have laid out altogether 6 times 5 beads or 6 times given a bead to each thread. The result will be the same, but the methods are different. 5 times 6 beads = 6 times 5 beads. This is simple, though $5 \times 6 = 6 \times 5$ may be difficult to explain.

We may take an oblong and rule it off in squares. When we examine its size, we can regard it either as 3 squares taken 4 times, counting the rows, or 4 squares taken 3 times, counting the columns. We may take a 12-inch strip of paper and measure it by a 6-inch strip. We shall find 12″ equal to 2 6-inch strips. Or we may measure it by means of a 2-inch strip and find 12″ equal to 6 2-inch strips. These examples need not be elaborated. They will suggest to the teacher a type of practical exercise to which the child's attention should be directed and also indicate the necessity for emphasising this principle as often as accident furnishes opportunity.

N.B.—In the rest of this book the knowledge of this fundamental principle of commutation of factors will be assumed.

Multiplication Tables.

1. For ease in construction, it is better that tables should have the form 2 times 9, 3 times 9, 4 times 9, instead of the familiar 9 times 2, 9 times 3, 9 times 4. Each is a condensed form of the same truth, but the first is the more natural expression of the result of adding 9's.

2. Tables should be constructed by concrete means, but the need for concrete aid diminishes. The first few should be constructed by each child, probably with more than one piece of apparatus. The next few can be built up by apparatus used by class as a whole; *e. g.* a large cardboard rectangle ruled off into rows of 7 squares for table of sevens. The last few need usually be constructed only by simple addition.

3. The tables, when constructed by children, should be entered in their notebooks. In the case of the earliest tables, they should be set simple little problems and allowed to solve them either by addition or by looking out the answer in their tables. The ease of this new plan will commend itself to them, so that they become ready to begin learning tables.

4. To lighten the labour of learning tables, various points may be noted—

(*a*) Make great use of the principle of commutation of factors. When a child knows up to 12 times 7, he knows the table of 8's as far as 7 times 8. Also in case of a combination being forgotten, its correlate should be emphasised.

(*b*) Pass continually from 8 times 7 to 7 times 7 and 9 times 7, so that if children forget one combination they know how to derive it from its neighbour above or below.

(*c*) The ease of the tables for 10's, 11's and 5's should be emphasised, as those children who most need the aid of knowing the tricks are least likely to discover them for themselves. Many grown-ups have never found out the help in remembering the table of 9's :—9, 18, 27, 36, 45 . . . , the unit figure diminishing by one each time. Also the sum of the digits is always 9 : $9 = 1 + 8 = 2 + 7 = 3 + 6 = 4 + 5$. These little things appeal to children's fancy, and, by obtaining greater attention without strain, help them to learn.

5. When all this has been said, every child must learn its tables thoroughly. This cannot be over-emphasised. Drill in tables should be prolonged and continuous. At a certain stage there is room even for such abstract drill sums as "Multiply 7169483 by 8"! Some form of reward or honour when a child has first mastered certain tables, or later, all its tables, would probably be of use; the test could be saying a table forwards, backwards, in any order, with, a little later, questions on division and on finding factors of a number. In the case of part of an upper standard proving weak in tables, the only remedy is by graded questions to discover the exact extent of the disease, then to build up the table with the children affected as far as may be necessary, and to have it thoroughly learnt and well practised, using all possible devices to ensure a complete mastery. Time lost on this will later be gain.

6. Children in upper standards will find it useful to know tables from 13's to 19's at least as far as products not exceeding 100. 14's and 16's are particularly useful in connection with avoirdupois weight. 15's can be connected with time measure, through ¼ hour.

Multiplication by Factors.

Suggestions as to a method which leaves room for children's initiative in learning this may not be amiss.

(*a*) Set question of type, "How many beads in 7 boxes each containing 43 beads?" followed by, "How many beads in 21 boxes, each containing 43 beads?"

Many will see how the answer to the first question can be used to answer the second. Continue setting pairs of similar questions until the idea of using an easy sum to do a harder one has taken root.

(*b*) Set question of type, "How many beads in 28 boxes, each containing 73 beads?" asking children to make up an easy sum which will help them to answer this harder one, and then work both.

The interest here is twofold: the interest of discovering a suitable simple sum, and the interest of finding that different discoveries lead to same ultimate result.

(*c*) This leads to discovery that any simple sum is suitable provided its multiplier goes exactly into given multiplier. Here introduce *factor* and give drill in finding factors of numbers.

(*d*) Give plenty of practice in multiplication by factors, having at first the meaning of each line briefly explained, but passing from easy to harder examples.

(*e*) Give special attention to such multipliers as 20, or 70, leading to doing the whole multiplication in one line, because multiplication by a 10-factor simply adds 0. This is essential preparation for long multiplication and should be extended to such multipliers as 700 or 4000.

Long Multiplication or *Multiplication by Partial Products.*

The idea underlying this is different from the idea underlying multiplication by factors. We take parts of the product and then combine them by addition. Again suggestions may be given for possible method of attack.

(*a*) Set a question of this type, "How many beads in 23 boxes, each holding 47 beads?"

Many children will find the correct answer for themselves, using a variety of methods—

(i) Beads in 22 boxes and add beads in 1 box.
(ii) Beads in 21 boxes and add beads in 2 boxes.
(iii) Beads in 20 boxes and add beads in 3 boxes.
(iv) Beads in 24 boxes and take away beads in 1 box.

Note that they are applying in each case the principle of partial products. These methods should be all placed on blackboard and compared. The favourites will be (i) and (iii), (i) because you multiply only by **2** and **11** and add, (iii) because you multiply by **20** in one line, multiply by **3** and add. The other two involve more work. Suggest looking for easiest method next time.

(*b*) Set such a question as, "How many beads in **53** boxes, each containing **94** beads?" Again a variety of methods will result, but there will be no doubt that the easiest is to take **50** boxes and **3** boxes and add. The setting down will in general be redundant at this stage.

e.g.

```
   beads                    beads
     9 4                      9 4
     5 0                        3
   -----                    -----
 4 7 0 0 in 50 boxes        2 8 2 in 3 boxes
   2 8 2
   -----
 4 9 8 2 in 53 boxes.
```

(*c*) Give several similar examples, until idea of multiplying by tens and units separately has taken root. Gradually let children criticise above method of setting down (or any similar one used by themselves) and lead them to the brief form—

```
   beads
     9 4
     5 3
   -----
 4 7 0 0 in 50 boxes
   2 8 2 in  3   „
 ---------
 4 9 8 2 in 53   „
```

Soon most children will dispense with full explanation.

(*d*) Extend to such multipliers as **123**.

(*e*) After some practice, they will see that to drop the 0's at end of line saves time; so they reach the simple rule of setting down first figure below multiplying figure and then moving each line one place to the right.

e.g.

```
      4 1 9
      3 1 2
    -------
  1 2 5 7
    4 1 9
      8 3 8
-----------
1 3 0 7 2 8
```

N.B.—Note that we here begin by multiplying by the left-hand figure instead of the more usual right-hand one. Those wishing to adhere to the other method can easily adapt to their views the setting down in (*b*) and (*c*) above. Reasons in favour of the course adopted here are—

(i) Children in finding out method for themselves reach this rule as they aim at getting the answer nearly right by their first multiplication.

(ii) It is of value in general to have the main part of result given first and the less important parts later.

(iii) If approximate methods in decimals are taught later, this order is necessary.

(iv) The one rule is to the child as simple and quick as the other, although less simple and quick to teachers brought up on a different one!

Short Methods of Multiplication.

Children's interest in short methods can easily be aroused. The following are typical instances—

1. To multiply by 25, add two 0's and divide by 4: $25 = 100 \div 4$.

2. To multiply by **125**, add three **0**'s, and divide by **8**:
$$125 = 1000 \div 8.$$
3. To multiply by $33\frac{1}{3}$, add two **0**'s, and divide by **3**:
$$33\tfrac{1}{3} = 100 \div 3.$$
4. To multiply by **99** or **999**, add two or three **0**'s and subtract the original number:
$$99 = 100 - 1 \text{ and } 999 = 1000 - 1.$$
5. To multiply by **427**, multiply by **7** and then add **60** times the first answer to it: for
$$427 = 7 + 420 = 7 + 60 \text{ times } 7.$$

Similarly to multiply by **963**, multiply by **900** and then add **7** times the first line, writing it as units instead of hundreds: for **963** = **900** + **63** = **9** hundreds + **7** times **9** units.

Checking of Multiplication Results.

Children should frequently correct their own work, by applying one or other of these methods—

1. Interchange multiplier and multiplicand and repeat multiplication.
2. Divide product by multiplier or multiplicand.
3. Use a different combination of partial products: *e. g.* **59** = **50** + **9** and also **59** = **60** — **1**, or **47** = **40** + **7** = **45** + **2** = **48** — **1**.

CHAPTER V

THE FUNDAMENTAL OPERATIONS—DIVISION

"*Multiplication is vexation,*
Division's twice as bad;
The rule of three it puzzles me,
And practice drives me mad."

To whatever extent modern methods have undermined the orthodoxy of this classic poem, one statement in it remains true; division is twice as bad as multiplication—twice as bad for teachers, twice as bad for pupils. We need to take all the comfort we can from the fact that in division we have to deal with an operation psychologically the hardest of the four; with methods of manipulation—in remainders and division by factors—comprehensible only by most careful introductory concrete presentation; with one method of manipulation—long division—so complex and teasing that only within the last three hundred years has it come into general use. Any one anxious to study previous methods of long division will find detailed accounts in Ball's *History of Mathematics* or Cajori's *History of Elementary Mathematics*. But all should remember that in teaching long division we are handling a method of manipulation which is extremely difficult. Probably one way of easing the difficulty is to postpone it as late as possible, until children's minds have developed more power to keep hold at one time of three distinct operations. It is a question whether it should ever be touched before Standard IV.; some of the most successful Arithmetic results are found in schools where it is postponed till the latter half of the Standard IV. period. In connection with it, one can then revise all tables of money, weights and measures, already learnt and used as far as division by factors.

We shall here begin by discussing the nature of division, paying special attention to the question of remainders. Later we shall deal in detail with the different steps to be taken in teaching division, when necessary outlining methods of attack which may prove suggestive to any finding this a difficult subject to handle.

The Nature of Division.

We looked at the multiplication operation from three different view-points: similarly we can look at division in three ways.

(*a*) Division may be regarded as bearing the same relation to subtraction that multiplication bears to addition. We ask, "How many times can I take 5 sticks from 17 sticks?" Here the relation to subtraction is obvious. From 17 take 5, another 5, another 5, and no more 5's are left to take. But just as multiplication involved the abstract idea of adding "so many times," so division involves the idea of subtracting "so many times." We think of taking away 3 times 5 at once instead of 5 + 5 + 5 until we can take no more away. The question then is, "How many times is subtraction possible?" Obviously this is harder than subtraction, because more abstract.

(*b*) Division also may be regarded as having the same relation to multiplication that subtraction bears to addition. Just as we ask, "How many must I add to 5 sticks to make 8 sticks?" so we ask, "How many times 5 sticks must I take to have 35 sticks?" In such a case division is simply inverse multiplication. When we ask "How many groups of 5 sticks must I take to have 37 sticks?" we have reached something more complex. We have inverse multiplication, with something else—subtraction. We may say that we have to take 7 groups of 5 sticks and 2 sticks besides, or, since 5 sticks form a group, 1 stick therefore $\frac{1}{5}$ group and 2 sticks $\frac{2}{5}$ group, we may say that we have to take $7\frac{2}{5}$ groups of 5 sticks. Each answer is correct; but to say that we have to take 7 groups of 5 sticks and $\frac{2}{5}$ stick besides, is wrong. In other words, in dividing 37 sticks by 5, our remainder is 2 sticks or $\frac{2}{5}$ group of 5 sticks but not $\frac{2}{5}$ stick. The simplicity of this illustration must be forgiven by those to whom it seems an easy matter. If such people will pause to count the number of ideas to be grasped by a child before answering correctly even such a simple question in division, they will no longer be surprised at their bad results. Obviously the fractional treatment of the remainder cannot be developed until children become familiar with fractional relations in general, *i.e.* in Standards III. and IV.

(*c*) Remembering the value of considering multiplication as making more complete the expression of a quantity measured in derived units, by measuring it in primary units, we reach another view-point for division, which leads us straight to the twofold nature of division. In division questions our quantity is always given as measured in primary units, *e.g.* 20 pints, 54 sticks, 13*s.* 4*d.* Sometimes we are told the size of the derived unit and asked to find how many such units are in the given quantity, *e.g.* how many 4-pint basins can I fill from 20 pints? How many yards at 3*s.* 4*d.* can I buy with 13*s.* 4*d.*? How many groups of 9 sticks can I make with 54 sticks? The answer in each case is an abstract number of units—5, 4, 6. The question asked provides the details—5 *basins*, 4 *yards*, 6 *groups*. But sometimes we are told how many derived units we are to divide the given quantity into and we have to find the size of the derived unit: *e.g.* I put 20 pints of milk into 5 basins: how much in each? I must buy 4 yards of cloth with 13*s.* 4*d.*: how much can I pay per yard? How many sticks will be in each group if I make 6 groups from 54 sticks? The answer in each case is a concrete quantity, the derived unit: 4 *pints*, 3*s.* 4*d.*, 9 *sticks*.

The connection of this twofold nature of division with the commutation of factors in multiplication is obvious. 20 sticks are 5 times 4 sticks or 4 times 5 sticks. Therefore, whether we divide 20 sticks by 4 sticks or by 4, the numerical part of our answer is 5. So that *mentally* the same operation of inverse multiplication or continued subtraction will give the correct result in both types of division.

The remainder difficulty is more serious. Let us consider simple examples.

(*a*) How many **4**-pint basins can I fill with **21** pints of milk?

21 pints $\div$ **4** pints gives **5**, but **1** pint is left over. Each basin holds **4** pints, so **1** pint is $\frac{1}{4}$ of a full basin. The answer therefore is: "I can fill **5** basins and leave **1** pint over," or "I can fill $5\frac{1}{4}$ basins." Both answers are correct, but "I can fill **5** basins and leave $\frac{1}{4}$ pint over" is wrong.

$$4 \,\underline{|\ 21 \text{ pints}\ }\qquad\qquad 4 \,\underline{|\ 21 \text{ pints}\ }$$
$$5 \text{ basins—}1 \text{ pint over.}\qquad\qquad 5\tfrac{1}{4} \text{ basins.}$$

(*b*) I wish to put **21** pints of milk into **5** basins; how much must I put in each?

21 pints $\div$ **5** gives **4** pints to each basin, but **1** pint is left over. This pint must either be left or a part of it must be put in each of the five vessels. That part, if all basins be treated alike, would be $\frac{1}{5}$. The result therefore is "**4** pints to each and **1** pint over," or "$4\frac{1}{5}$ pints to each," but "**4** pints to each and $\frac{1}{5}$ pint over" is wrong.

$$5 \,\underline{|\ 21 \text{ pints}\ }\qquad\qquad 5 \,\underline{|\ 21 \text{ pints}\ }$$
$$4 \text{ pints—}1 \text{ pint over.}\qquad\qquad 4\tfrac{1}{5} \text{ pints.}$$

In this case the second result makes better sense as an answer to the question asked.

We note that both in (*a*) and (*b*) the *mental* operation is the same and the rules for dealing with a remainder are the same.

It may be well now to add to our simple discussion the more technical terms occurring in this connection. In the first case, when we divide a concrete quantity by a concrete quantity we are *measuring* a quantity by a new unit; *dividing* a quantity into smaller quantities, asking the question "how many times" (Latin *quot*) one quantity is contained in another. So this aspect of division is called *measurement* or *quotition* or *division* properly speaking. In the second case, when we divide a concrete quantity by an abstract number, we are *sharing* that quantity amongst a given number of people and finding the share of each, or *parting* it amongst a certain number. This is *sharing* or *partition.* In the first case the result is an abstract number of times; in the second, a concrete quantity.

Fortunately for children, the labour of learning division is not twofold. We have seen that the same *mental* activity is evoked by both types of question. The practical necessities arising out of the twofold nature of division are—

(i) That our children early have an intuitive understanding of the principle of commutation of factors.

(ii) That in early work with easy numbers we avoid the word "divide," as is done in above examples. "Division" is a good name for the one mental activity (inverse of multiplication) behind both. Numerous examples of both must be given from the earliest stages until children instinctively perform the mental operation of division at the call of either type of question. Insistence on their result being an answer to the given question prevents blunders: *e. g.* children can be made to laugh at themselves for saying "7 pints" in answer to "How many basins?" "Measuring" and "sharing" are useful terms in the earlier stages of teaching division.

(iii) That in handling the more complex stages of written division—*i. e.* division by factors and long division—we choose at first one of the two types for our concrete illustrations and, in our explanations of remainders and notation, consistently refer to the type selected. In other words, the full explanation of a harder division sum requires very careful preparation of its concrete embodiment if the result is not to be confusion. Having developed the new method of manipulation from one of the two types, we can use it as an instrument for tackling any question requiring the mental operation termed "division."

Division by Numbers not exceeding 12.

The earliest stages of division have already been sufficiently indicated. Put briefly, what is necessary is to teach division by oral work along with multiplication tables, taking apparatus until children use inverse multiplication and subtraction to obtain the answers to both "quotition" and "partition" questions. At this stage, remainders must be treated as whole numbers left over; later, when the general notation for fractions is introduced, about Standard III. or IV., remainders can be treated from the fractional point of view.

When we advance to written work, we reach the problem of teaching children to divide numbers beyond reach of multiplication tables by divisors not exceeding 12, teaching what most term "short division." It is well to note at once that "short division" and "long division" are not radically different division methods, but a short and a long way of writing out the same problem.

e. g. I have to share 873 sticks amongst 7 people; how many sticks can each person have?

(*a*) LONG DIVISION.

```
  1 2 4
  ─────
  H.T.U.
7)8 7 3(1 H
  7
  ─────
7)1 7 ten-bundles(2 T
  1 4     ,,
  ──────
7)  3 3 sticks(4 U
    2 8   ,,
  ──────
      5   ,,
```

(1) Share 8 hundred-bundles of sticks amongst 7 people. 1 *H to each* and 1 H over.

(2) Share 1 H and 7 ten-bundles, *i. e.* 17 T amongst 7 people. 2 *T to each* use 14 T, leaving 3 T over.

(3) Share 3 T and 3 single sticks, *i. e.* 33 U amongst 7 people. 4 *U to each* use 28 U, leaving 5 U over.

Altogether 1 H, 2 T, 4 U to each of 7 people, 5 sticks over, *i. e.* 124 sticks to each of 7 people, with 5 sticks over; or $124\frac{5}{7}$ to each of 7 people, since each stick over allows $\frac{1}{7}$ stick to each person.

We have here written out the quotient step by step in H's, T's and U's; but all that is necessary is to write each above the dividend in its proper position, as is indicated in above example. The use of coloured chalk to distinguish H's, T's and U's is an aid to beginners in division.

(*b*) Short Division.

$$\begin{array}{rl} & \text{H.T.U.} \\ 7 \,|\, & \underline{8\ 7\ 3} \text{ sticks for 7 men} \\ & 1\ 2\ 4 \quad ,, \quad ,, \ 1 \text{ man and 5 sticks over} \\ \text{or} & 1\ 2\ 4\tfrac{5}{7} \quad ,, \quad ,, \ 1 \text{ man.} \end{array}$$

(1) Share 8 H amongst 7 people. 1 *H to each.* 1 H over.

(2) Share 1 H + 7 T or 17 T amongst 7 people. 2 *T to each.* 3 T over.

(3) Share 3 T + 3 U or 33 U amongst 7 people. 4 *U to each.* 5 U over.

∴ 124 to each and 5 over.

Obviously the difference between (*a*) and (*b*) is that in (*a*) we write out fully the multiplication required at each stage (7×1, 7×2, 7×4) and also the remainders of each stage, 1H, 3T, before reducing them from H's to T's, and from T's to U's. When divisors are large, we cannot do multiplication or subtraction mentally and must use the long way of writing our working down; when divisors are small, we can do the working mentally and need write down only the result.

This brings us to a difference of method in teaching written division. Some teachers of young children teach the long-division form of writing down division with small divisors, and then later the short-division form as a contraction; others teach the short-division form at once for small divisors, and later, when divisors are larger and short-division is insufficient, they introduce the long way of setting down sums already set down shortly. We wish here to advocate that the short-division form be taught before the long-division form, and to meet the objections of those advocating the other.

(1) Such say that the long-division form, showing all the operations involved, is easier for little children than the short-division form. To this two answers must be

given. First, the less writing necessary, the clearer is the operation for little children, with whom writing hides rather than reveals the truth. Next, the short-division method is quite easy for them, provided that examples are properly chosen, that simple apparatus is used, and that the oral work in earlier stages has been thoroughly taught. Before touching written division, children should be able to answer orally questions on table used (*e. g.* table of 5's) as hard as : "I have to share 33 sticks amongst 5 people. How many can each have; how many will be left over?" Then if convenient concrete embodiment, such as sticks or tablets, be given to questions of these successive types, no great difficulty is found.

(i) No remainders at any stage, but H's, T's, U's taken in turns—
e. g. $63 \div 3$, $848 \div 4$

(ii) Remainders only at end—
e. g. $98 \div 3$, $849 \div 4$

(iii) Remainders only at beginning and end—
e. g. $158 \div 3$, $249 \div 4$

(iv) Remainders only in middle—
e. g. $84 \div 3$, $892 \div 4$

(v) Remainders everywhere—
e. g. $77 \div 4$, $765 \div 3$.

Children will proudly make the journey for themselves if it is subdivided into such easy stages. Long division can only be demonstrated to them; no need for it can possibly be felt at this stage.

(2) Those in favour of teaching the long-division form first, argue that it will help later with teaching of long division. But so great a period of time usually intervenes between these two stages that the necessary revision of the long-division way of writing down a sum with an easy divisor is very much like teaching it afresh. All that is necessary seems to be that children, learning long division in Standard III., or, preferably, Standard IV., should first practise the method as a long way of setting down sums already worked by short division, in order to advance to ability to deal with divisors now too difficult

for them. Here they can appreciate the value of the longer form.

Thus we maintain that the short-division form of setting down division sums should come first. The history of Arithmetic, showing the early development of the short-division form and the very late development of our present long-division form, supports our contention.

In conclusion, it is worth while to note that the common use of + in setting down remainders in short division is fundamentally unsound—

$$i.e. \quad 3\,\underline{|\,1\;4\;9}$$
$$\quad\quad 4\;9 + 2$$

For the **49** are groups of **3** and the **2** are single units. "**49** and **2** over" is correct but slow to write; "and" can be contracted to "&." "**49** ,, **2**" is also possible as a final contraction, since ",," is used to separate different units in dealing with money, weights and other measures. But **49** + **2** can only be equivalent to **51**.

Division by Factors.

The idea underlying this method of division is different from that underlying long and short division. As in multiplication by factors, this is a case of answering first an easy question in order to be able to answer a harder one; but the greater complexity of division and the trouble concerning remainders make division by factors more difficult than multiplication by factors. We shall here outline a possible method of approach.

At first it is wisest to work very simple examples by the aid of apparatus, at the same time writing down the full explanation step by step; sticks, matches and tram tickets are always convenient. In suggesting examples, we therefore assume at each stage, first, the use of apparatus in answering questions, and, along with this, a full explanation in writing; next, working of a few similar examples in writing and testing the results by apparatus; third, the written solution of miscellaneous problems involving the degree of manipulative skill acquired at the stage in question. Obviously the choice of larger dividends, and divisors containing harder factors, and of problems embodying both "quotition" and "partition" will increase the difficulty to any degree required.

(*a*) Examples with no remainder difficulty, to teach method of answering a harder question by first answering a suitable easier one.

e.g. (1) I put 75 sticks in groups of 3 sticks; how many groups?

$$\begin{array}{r|l} 3 & 7\ 5 \text{ sticks} \\ \hline & 2\ 5 \text{ groups of } 3. \end{array}$$

(2) I put 75 sticks in groups of 15 sticks; how many groups?

With apparatus it is easy to see that to obtain groups of 15, we use 5 groups of 3—

$$\begin{array}{r|l} 5 & 2\ 5 \text{ groups of } 3 \\ \hline & \quad 5 \text{ groups of } 15. \end{array}$$

After one or two examples children will be able to suggest easy questions helpful in answering the harder question set, discovering that different easy ones can be chosen without affecting the final result, provided only that the first divisor be a factor of the second divisor. Compare multiplication by factors.

In setting down on blackboard, it is helpful to use different colours to distinguish between single sticks, groups of 3 and groups of 15.

(*b*) A few examples introducing remainder only in second part.

e.g. I put 72 sticks in groups of 16; how many groups?

$$1\ 6\left\{\begin{array}{r|l} 2 & 7\ 2 \text{ sticks} \\ \hline 8 & 3\ 6 \text{ groups of } 2 \\ \hline & \quad 4 \text{ groups of } 16 \text{ ,, } 4 \text{ groups of } 2 \end{array}\right\}\ \begin{array}{l} 4 \times 2 = 8 \\ \quad \text{over.} \end{array}$$

(*c*) Examples involving remainders in both parts.

e.g. $78 \div 14$

$$1\ 4\left\{\begin{array}{r|l} 7 & 7\ 8 \text{ sticks} \\ \hline 2 & 1\ 1 \text{ groups of } 7 \text{ ,, } 1 \text{ stick} \\ \hline & \quad 5 \text{ groups of } 14 \text{ ,, } 1 \text{ group of } 7 \end{array}\right.\ \left\{\begin{array}{l} 1 \times 7 + 1 \\ = 8 \text{ over} \end{array}\right.$$

If coloured chalks are used and the remainders are written in the correct colour, the rule for finding the total remainder stands out very clearly.

(*d*) With older children, complicated divisors to test mastery of process.

e.g. Divide 4879 by 135, using factor method.

$$135 = 3 \times 5 \times 9$$

$$135 \left\{\begin{array}{r|l} 9 & 4\ 8\ 7\ 9 \\ 5 & \quad 5\ 4\ 2 \text{ groups of } 9\ ,,\ 1 \\ 3 & \quad 1\ 0\ 8 \quad ,, \quad 45\ ,,\ 2 \text{ groups of } 9 \\ & \qquad 3\ 6 \quad ,, \quad 135\ ,,\ 0 \quad ,, \quad 45 \end{array}\right\} \begin{array}{r} 0 \times 45 \\ +\ 2 \times 9 \\ +\ 1 = 19 \\ \text{over.} \end{array}$$

Long Division.

It is unnecessary to repeat in any detail what has already been discussed in connection with short division. We may summarise by stating again that long division is a late development of the race, never easy for children and best postponed as late as possible; that, like short division, long division is based on dividing each of the units of notation in turn and reducing the remainders at each stage to the next lower unit; that long division is simply the setting down in full of the work done mentally in short division and is best practised as a method by first writing out in full sums which can be worked by short division. Basing our suggestions on this summary, we outline a possible development.

(*a*) Set a problem which children cannot answer by short division or division by factors, *e.g.* one involving the division of 847 by 29. Children, being puzzled, will respond to the suggestion to study short division more closely in order to devise a plan for dealing with hard divisors.

(*b*) Practise working long-division questions with divisors not exceeding 12, including ones involving a zero in the quotient. The result should be placed above the dividend, to emphasise place value and show connection with short-division results.

e.g. a problem involving 11937 ÷ 7.

```
      1 7 0 5
   ___________
  7)1 1 9 3 7 yards
      7
      ___
      4 9
      4 9
      _____
          3 7
          3 5
          ___
            2
```

$1705\frac{2}{7}$ yards to each or 1705 yards to each and 2 yards over.

(*c*) Practise long-division method with such divisors as 20, 30 or 70.

(*d*) Practise with divisors like 31, 43, 61, which are approximately 30, 40, 60. Here value of quotient is easily seen.

(*e*) Practise with divisors like 19, 28, 37, which are approximately 20, 30, 40. Here it is a little more difficult to see value of quotient.

(*f*) Practise with such divisors as 26, 34, 65, where it is often difficult to see quotient because of carrying figure.

At each stage work should include examples without and with remainders and examples giving a zero in quotient. Children can obtain double practice from same problem by repeating division, using first quotient as their second divisor; their first divisor will be their new answer. The passage from 2-figure divisors to larger ones does not usually give much difficulty, if children have become competent with 2-figure divisors.

Italian Method for Long Division.

In this method for long division, the results of multiplication are omitted, the remainders after subtraction at each stage only being set down. This is not hard to do if subtraction has been taught as inverse or complementary addition. For an example worked in full see the section in Chapter III. on "Subtraction as Complementary Addition" (p. 53).

Checking of Division Results.

Children should frequently correct their own work, by applying one or other of these two methods—

(1) Multiply quotient by divisor, add remainder to product; result should be the dividend.

(2) Divide dividend by quotient; result should be the original divisor, with the same remainder as in question set.

CHAPTER VI

APPLICATION OF THE FUNDAMENTAL OPERATIONS TO MONEY

In this chapter we propose briefly to consider the natural background to children's work with money and its close connection with their lives, leaving the working out of the suggestions to the individual teacher. Here, if anywhere, the variety in children's experience of life must be taken into account; boys and girls, town and country children, well-to-do and slum children, the same children at different ages, all will regard money in different ways. The one fundamental necessity is to link the learning of sums dealing with the manipulation of money to the life experience of our pupils, whatever it be. After emphasising this by illustrations and suggestions, we propose to deal in detail with the compound rules for money, since such a full treatment will obviate the necessity of discussing them for weights and measures.

The Place of Money in Life.

On this subject we all have views. Probably the best preparation for teaching "money" in any standard is to think out the reasons for the importance of money, our own problems with regard to earning, spending and saving, and the extent to which we find it necessary and difficult to perform calculations involving money. It is easy to make this subject "live" to ourselves and so to prepare ourselves to make it "live" to our pupils. In this case we may learn as well as teach, provided only that by sympathetic treatment we can reach the actualities of our pupils' experiences with money.

With little children undoubtedly the first requisite is a good supply of coins. Some will be fortunate enough to have at their disposal sufficient cardboard coins to give each child his own purse. In other schools, children can make coins in handwork lessons. In our young days we have all experienced the joy of scribbling with pencil over a paper resting on an embossed bookcover and discovering the pattern of the book reproduced in our pencil-shading. A piece of stout paper used in this way will give a picture of one side of any coin; if cut out and then placed against other side of coin, with clean side uppermost, the paper will have on this second side the second side of the coin. The delay in making and the lack of colour and durability make these coins inferior to cardboard ones, but many will find them valuable.

Undoubtedly the best natural background for younger children's money work is shopping with these "make-believe" coins. Many of the goods required for selling can be made by children in handwork lessons or out of school; much is of a very simple character, such as sand, pebbles, strips of newspaper gummed together and rolled. Some of it can be obtained from manufacturers and shopkeepers: *e.g.* many manufacturers will send a supply of show packets, or drapers will give the rolls of paper used inside rolls of ribbon. The shopping can be so arranged as to teach definite stages: *e.g.* addition of pence only, or of pence and halfpence; subtraction of pence and halfpence from **1***s*. or **2***s*., or of shillings, pence and halfpence from **10***s*.; multiplication of shillings and pence; measuring in yards, $\frac{1}{2}$'s, $\frac{1}{4}$'s and $\frac{1}{8}$'s of a yard; weighing in lb. and oz.; the making out of bills; keeping of accounts; the relation of goods sold to cash taken in a shop; prices of common articles; prices of oz., of $\frac{1}{2}$ and $\frac{1}{4}$ yards, if price of lb. or yard is given; wise expenditure of a given sum of money to accomplish a given end. For instance, to teach addition of pence and halfpence and subtraction from **2***s*., objects would be labelled from **1½***d*. to **11½***d*. and **2***s*. would be given to each child to spend, practice becoming more difficult as children were required to buy several articles. Or, to teach measuring and weighing, children could be sent to various shops, one selling "make-believe" ribbon by the yard, another sugar or tea by the lb., and a third coloured water by the pint. To teach halving and quartering of sums of money, the same shops could be used, children being instructed to buy $\frac{1}{2}$ or $\frac{1}{4}$ lb., yard, or pint. Needless to say, the most important type of practice is given when all these different operations are combined and children can deal with each as it arises.

Shopping lessons are difficult to take successfully because of the trouble involved in keeping every one busy. Apart from the occasional lesson with one shop for the whole class, children taking turns as shopkeepers, accountants and purchasers, while the class counts cost of purchases and "change" required—a type of lesson always useful as an introduction, and often, owing to crowded rooms, the only type possible—probably the most useful arrangement is either to have one good shop for the school and let classes take it in turns to have a day's shopping, or, better still, in standards which require much shopping work, to have a set of, say, 8 shops made, by putting desks or tables against the walls round the room. In this case, with a class of 40 children, 16 would be occupied in these shops, 8 as salesmen, and 8 to keep the accounts and check the day's sales, *i.e.* make sure that money taken equals value of goods sold, and that money taken added to money previously in till, equals cash in hand at end of day. The remaining 24 would be kept busy planning their expenditure, purchasing their goods and keeping their accounts, it being good to allow them to work in pairs, one talking and the other writing. The purchasers' accounts should show that money spent added to cash in hand at end of day equals money originally in hand. It is surprising how much the children do in keeping each other accurate, and how little the teacher need do in the way of definite "teaching." The written accounts need to be overlooked, while good manners for purchasers and salesmen can be naturally inculcated. With larger classes than 40, it is perhaps best to develop the plan of letting sections of the class do shopping in turns while the others have practice work in written Arithmetic.

In connection, then, with shopping, many of the necessary money operations can be practised in our lower standards, with the great advantage that the children's interest is naturally evoked and so the needed attention and concentration are secured without undue strain. Time spent on setting up the shops and planning introductory lessons will thus later be saved; willing minds learn quickly. In addition, we can make Arithmetic seem "worth while" by thus linking it closely with children's lives and natural desires, and we can teach along with number work many other useful lessons, ranging from good manners to wise expenditure. A slightly different method for occasional use is to correlate our Arithmetic with the Geography of the district, and in connection with imaginary journeys taken by rail, set up a ticket-office and arrange for buying railway tickets; singles and returns, wholes and halves, excursion and market tickets lend variety and increase the difficulty of the necessary calculations.

With older children the actualities of coins and shops can be dropped, provided that we give them problems or invite them to raise problems which embody their daily experience of money values. First of all, prices of common articles must be taught, with some discussion of the circumstances in which cheapness may be false economy. The gain in buying large instead of small quantities can be touched on, if notice is taken of the real difficulty of storing a quantity of perishable goods, especially under certain serious housing conditions. We can refer to butchers' and fishmongers' common practice of cheating their customers by charging too much for fractions of a pound, unless their arithmetic is found equal to detecting the error. Given a certain sum of money, girls can plan a set of new clothing for themselves or the furnishing of their home when married. The cost of building, roofing and decorating houses provides money work along with mensuration. We can touch on the difficulty of planning one's expenditure on fluctuating wages, applying the theory of averages. We can drive home the extravagance of the "payment by instalment" system, if children calculate how much is actually so paid for some definite article whose real cost price is known; for instance, sewing machines cost about twice as much, if paid for in this way, and purchasers often forfeit them at the end, because unable to continue paying their instalments, thus losing everything. It is usually wiser and much cheaper to save and buy second-hand. It is good for girls at some stage to spend one lesson a week in planning a weekly expenditure for their imaginary families on their husbands' weekly wages; the teacher can invent emergencies to tax the ingenuity of older girls, *e.g.* A's illness, or B's need of new boots.

Other illustrations, in connection with pawnbroking, payment of rent, rates and taxes, and methods of saving, we postpone till dealing with percentages and interest, as they involve more difficult manipulative methods than are assumed at this stage. But the above material provides a good background for rapid addition, subtraction and multiplication of money, and for calculation of prices by short methods, or of prices involving difficult fractions. Reduction comes into all other money operations, but problems connected with the gathering of funds to provide a trip or raise some subscription would introduce this more definitely, and also division. The alert and sympathetic teacher will find some point of contact with his pupils' real life, and, contact once established, will be surprised at the ease with which the subject develops when children appreciate its value and begin to raise problems and make up their own examples. Arithmetic here comes so close to lessons on citizenship and home-management that definite correlation is easy.

Assuming that some such natural background is given to children's money work throughout the school, we pass on to treat the various compound rules in detail, noting—

(1) that these methods of manipulation should be first taught in connection with some problem of real interest;
(2) that they should later be practised mechanically until at each stage speed and accuracy are ensured;
(3) that throughout the course they should be used not only mechanically, but as methods necessary for dealing with interesting problems.

If this threefold treatment be kept in view, children will learn to use money sensibly and will also become quick and accurate in mental calculations. Both lessons are essential as preparation for life in this world. In what follows we shall merely discuss the skeletons of the various rules, omitting further reference to the flesh and blood which must clothe them before they are presented to our pupils.

Compound Addition.

The main difficulty in acquiring skill lies in changing pence to shillings, and, to a less degree, shillings to pounds. In adding any column of figures, two methods may be adopted—

1. The total number of, say, pence may be found and the pence then changed to shillings, either—

(*a*) by use of the so-called pence table; or,
(*b*) by use of multiplication table for 12's.

2. The pence may be made up to shillings during the addition, so that number of pence at any stage never exceeds 11.

Examples.

I. Find value of $7d. + 6d. + 8d. + 4d. + 10d. + 8d.$

1. Total = 43 pence, by simple addition.

(*a*) Pence table says 40 pence = 3*s*. 4*d*.
$\therefore$ 43 pence = 3*s*. 7*d*.

(*b*) Table of 12's tells us that there are 3 times 12 pence in 43 pence and 7*d*. over.
$\therefore$ 43 pence = 3*s*. 7*d*.

2. $7d. + 6d. = 7d. + 5d. + 1d. = 1s.\ 1d.$; $1s.\ 1d. + 8d. = 1s.\ 9d.$; $1s.\ 9d. + 4d. = 2s.\ 1d.$; $2s.\ 1d. + 10d. = 2s.\ 11d.$; $2s.\ 11d. + 8d. = 3s.\ 7d.$

II. Find sum of $7\frac{1}{2}d.$, $6\frac{3}{4}d.$, $8\frac{1}{2}d.$, $4\frac{1}{4}d.$, $10\frac{1}{2}d.$

1. Total no. of farthings = 10 = $2\frac{1}{2}d.$
Carry 2*d*.
Total no. of pence = 37*d*. = 3*s*. 1*d*., by (*a*) or (*b*) above.
$\therefore$ Total = $3s.\ 1\frac{1}{2}d.$

2. $7\frac{1}{2}d. + 6\frac{3}{4}d. = 1s.\ 1\frac{1}{2}d. + \frac{3}{4}d. = 1s.\ 2\frac{1}{4}d.$; $1s.\ 2\frac{1}{4}d. + 8\frac{1}{2}d. = 1s.\ 10\frac{3}{4}d.$; $1s.\ 10\frac{3}{4}d. + 4\frac{1}{4}d. = 1s.\ 11d. + 4d. = 2s.\ 3d.$; $2s.\ 3d. + 10\frac{1}{2}d. = 3s.\ 1\frac{1}{2}d.$

III. Find the sum of 11*s*., 15*s*., 6*s*., 13*s*., 7*s*.

1. Total = 52 shillings = £2 12*s*., dividing by 20.

2. 13*s*. + 7*s*. = £1; £1 + 6*s*. + 15*s*. = £1 + £1 + 1*s*. = £2 1*s*.; £2 1*s*. + 11*s*. = £2 12*s*.

Comments.

(*a*) Obviously, in adult life, sometimes 1 is easier, sometimes 2, and we can check our work by using both methods in turn. Therefore, at some stage, children should use both.

(*b*) The difficulty of changing pence to shillings must be taken in muliplication even if evaded in addition; therefore, at a comparatively early stage, children should use both methods.

(*c*) Children begin to add money before they are sure of table of 12's; hence arises the common practice of making them learn by heart the pence table—

12*d*. = 1*s*.
20*d*. = 1*s*. 8*d*.
24*d*. = 2*s*.
30*d*. = 2*s*. 6*d*., and so on.

This is used instead of table of 12's. The result is a lack of accuracy and speed owing to confusion between *tens* of pence and *shillings* of pence. Two plans may be adopted to obviate this difficulty in teaching addition of money before table of 12's is sound.

(i) A diagram of 12's may be constructed and used by children until table of 12's becomes familiar. Part of a suitable form is suggested—

	1	2	3	4	5	6	7	8	9	10	11	12	1*s*.
1*s*.	13	14	15	16	17	18	19	20	21	22	23	24	2*s*.
2*s*.	25	26	27	28	29	30	31	32	33	34	35	36	3*s*.
3*s*.	37	38	39	40	41	42	43	44	45	46	47	48	4*s*.

(ii) Until table of **12**'s is known, children can do addition by making up pence to shillings, this operation depending only on the analysis of **12**.

Either method should give better results than the use of the pence table. There is no reason why both should not be adopted, children learning as early as possible thus to check their own work and finding out for themselves on which method a given question can be worked most quickly by individual or by class.

Compound Subtraction.

The only remaining point worthy of note is that each method of subtraction already dealt with (see Part II. Chapter III.) can be used in Compound Rules.

EXAMPLE.—Find the value of **£4 13**s. **7**d. – **£2 15**s. **3**d.

I. *Decomposition.*

(*a*) Before subtraction—

$$\left.\begin{array}{l}\pounds 4\ 13s.\ 7d.\\ \pounds 2\ 15s.\ 8d.\end{array}\right\}=\left\{\begin{array}{llr}\pounds 4 & 12s. & (12+7)d.\\ \pounds 2 & 15s. & 8\ d.\\ \hline & & 11\ d.\end{array}\right.=\left\{\begin{array}{lcr}\pounds 3 & (20+12)s. & (12+7)d.\\ \pounds 2 & 15\ s. & 8\ d.\\ \hline \pounds 1 & 17\ s. & 11\ d.\end{array}\right.$$

(*b*) After subtraction—

$$\left.\begin{array}{l}\pounds 4\ 13s.\ 7d.\\ \pounds 2\ 15s.\ 8d.\end{array}\right\}=\left\{\begin{array}{lcr}\pounds & (20+13)s. & 7d.\\ & 15\ s. & 8d.\\ \hline \pounds 1 & s & d.\end{array}\right.=\left\{\begin{array}{lcr}(\pounds & s. & (12+7)d.\\ & s. & 8\ d.\\ \hline \pounds 1 & 17\ s. & 11\ d.\end{array}\right.$$

II. *Equal Additions*—

$$\left.\begin{array}{l}\pounds 4\ 13s.\ 7d.\\ \pounds 2\ 15s.\ 8d.\end{array}\right\}=\left\{\begin{array}{llr}\pounds 4 & 13s. & (12+7)d.\\ \pounds 2 & 16s. & 8\ d.\\ \hline & & 11\ d.\end{array}\right.=\left\{\begin{array}{lcr}\pounds 4 & (20+13)s. & (12+7)d.\\ \pounds 3 & 16\ s. & 8\ d.\\ \hline \pounds 1 & 17\ s. & 11\ d.\end{array}\right.$$

III. *Complementary Addition*—

	(1) *Pence.*
£4 13*s*. 7*d*.	8 + ? = 7 is impossible.
£2 15*s*. 8*d*.	8 + ? = 1*s*. 7*d*.; ? = 4*d*. + 7*d*. = 11*d*.
£1 17*s*. 11*d*.	Carry 1*s*. from 1*s*. 7*d*.
	(2) *Shillings.*
	1 + 15 + ? = 13 is impossible.
	16*s*. + ? = £1 13*s*.; ? = 4*s*. + 13*s*. = 17*s*.
	Carry £1 from £1 13*s*.
	(3) *Pounds.*
	1 + 2 + ? = 4; ? = £1.
	∴ Add on £1 17*s*. 11*d*. to give £4 13*s*. 7*d*.

Compound Multiplication.

Multipliers less than 12 or easily factorised give no difficulties beyond those already noted in Chapter IV concerning multiplication by factors and earlier in this chapter concerning the change of pence to shillings and shillings to pounds. Weak work in these rules is usually due to a variety of definite reasons, as was suggested on p. 15.

Calculation of prices introduces short methods of multiplication of money; a few typical cases may be given as suggestions.

(i) (3*s*. $11\frac{1}{2}$*d*.) × 7 = 7 × 4*s*.—7 × $\frac{1}{2}$*d*. = 28*s*. − $3\frac{1}{2}$*d*. = 27*s*. $8\frac{1}{2}$*d*.

(ii) (19*s*. 5*d*.) × 32 = 32 × £1 − 32 × 7*d*. = £32 − 32 × 6*d*. − 32*d*. = £32 − 18*s*. 8*d*. = £31 1*s*. 4*d*.

(iii) 7 × (1*s*. $4\frac{1}{2}$*d*.) + 7 × (3*s*. $7\frac{1}{2}$*d*.) = 7 times (1*s*. $4\frac{1}{2}$*d*. + 3*s*. $7\frac{1}{2}$*d*.) = 7 × 5*s*. = £1 15*s*.

(iv) 9 × (3*s*. 6*d*.) + 14 × (1*s*. 9*d*.) = 9 × (3*s*. 6*d*.) + 7 × (3*s*. 6*d*.) = 16 × (3*s*. 6*d*.) = 48*s*. + 8*s*. = 56*s*.

(v) 80 × (7*s*. $4\frac{1}{2}$*d*.) + 81 × (2*s*. $7\frac{1}{2}$*d*.) = 80 (7*s*. $4\frac{1}{2}$d. + 2*s*. $7\frac{1}{2}$*d*.) + (2*s*. $7\frac{1}{2}$*d*.) = 80 × 10*s*. + (2*s*. $7\frac{1}{2}$*d*.) = £40 2*s*. $7\frac{1}{2}$*d*.

(vi) 240 × (3*s*. $1\frac{1}{2}$*d*.) = $37\frac{1}{2}$*d*. × 240 = $37\frac{1}{2}$ × 240*d*. = $37\frac{1}{2}$ × £1 = £37 10*s*.

(vii) $238 \times (4s.\ 2\frac{1}{4}d.) = 50\frac{1}{4}d. \times 240 - (4s.\ 2\frac{1}{2}d.) \times 2$ $= £50\ 5s. - 8s.\ 5d. = £49\ 16s.\ 7d.$

(viii) $7\frac{1}{2}$ lb. at $9\frac{1}{2}d. = 7 \times 9d. + \frac{7+9}{2}d. + \frac{1}{4}d. =$ $(63 + 8\frac{1}{4})d. = 5s.\ 11\frac{1}{4}d.$

Or $11\frac{1}{2}$ yards at $2\frac{1}{2}d. = 11 \times 2d. + \frac{11+2}{2}d. + \frac{1}{4}d. =$ $(22 + 5\frac{3}{4})d. = 2s.\ 3\frac{3}{4}d.$

The development of the dozens rule may be shown in more detail as its applications can be far-reaching—

(i) $12 \times 3d. = 3s.$, $12 \times (2s.\ 5d.) = 12 \times 29d. = 29 \times 12d. = 29s.$

(ii) $12 \times \frac{1}{2}d. = 6d.$, $12 \times \frac{1}{4}d. = 3d.$, $12 \times \frac{3}{4}d. = 9d.$ $\therefore 12 \times (2s.\ 7\frac{1}{2}d.) = 12 \times 31\frac{1}{2}d. = 31s.\ 6d.$

(iii) $36 \times (3s.\ 2\frac{1}{4}d.) = 3 \times 12 \times (3s.\ 2\frac{1}{4}d.) = 3 \times (38s.\ 3d.) = £5\ 14s.\ 9d.$

(iv) $47 \times 7\frac{1}{2}d. = 48 \times 7\frac{1}{2}d. - 7\frac{1}{2}d. = 30s. - 7\frac{1}{2}d. = 29s.\ 4\frac{1}{2}d.$

(v) $34 \times (6s.\ 2\frac{3}{4}d.) = 36 \times (6s.\ 2\frac{3}{4}d.) - 2 \times (6s.\ 2\frac{3}{4}d.)$ $= 3 \times £3\ 14s.\ 9d. - 12s.\ 5\frac{1}{2}d.$

Multipliers such as 37 or 259, not to be factorised, lead to a variety of methods. We may work the first by taking 6 times the multiplicand, 6 times this product, and add to it the multiplicand: that is $37 = 6 \times 6 + 1$. Or we may multiply by 10, then this product by 3 and add to it the multiplicand multiplied by 7: that is $37 = 10 \times 3 + 7$. In the second case we may take 240 times the given sum, applying the short method of calling each penny £1; then add to this product 19 times the given sum as indicated for multiplication by 37, for $259 = 240 + 19$. Or we may take 10 times the multiplicand, and 10 times this product; twice this second product, together with 5 times the first product and 9 times the multiplicand, is the required result, for $259 = 10 \times 10 \times 2 + 10 \times 5 + 9$. These methods are all based on the ideas of multiplication by factors and by partial products, the two ideas being combined as may seem most convenient in any given case.

A totally different plan is to extend the short-multiplication method to large multipliers, a method frequently called "long multiplication"—

EXAMPLE—

£5 17s. 9½d. × 37	(i)	37 × ½d. = 18½d.	Set down ½d.
37	(ii)	18d.	& carry 18d.
£217 18s. 3½d.		9 × 37d. = 333d.	
		351d. = 29s. 3d.	
	(iii)	29s.	Set down 3d. & carry 29s.
		10 × 37s. = 370s.	
		7 × 37s. = 259s.	
		658s. = £32 18s.	
	(iv)	£32	Set down 18s. & carry £32.
		£5 × 37 = £185	
		£217	

Yet a third plan is to use the method commonly called "Practice" or "Method of Aliquot Parts."

EXAMPLE.—£5 17s. 9½d. × 259

£259	259 × £1	or, using decimals	£259
£1295	259 × £5	£5	1295
129 10s.	259 × 10s.	10s.	129·5
64 15s.	259 × 5s.	5s.	64·75
32 7s. 6d.	259 × 2s. 6d.	2s. 6d.	32·375
3 4s. 9d.	259 × 3d.	3d.	3·2375
10s. 9½d.	259 × ½d.	½d.	·5396
£1525 8s. 0½d.	259 × £5 17s. 9½d.		1525·402*

Result = £1525 8s. 0½d.

* Neglecting unnecessary figures.

This third method is learnt by every child at some stage, but too frequently its real use as a short method for calculating prices is obscured by unnecessary explanations and by the awkwardness of division of money. If decimals are used, it is certainly the quickest way of dealing with multiplication of money by large multipliers. In case of easier multipliers, the first method generally gives better results than the method we have called "long multiplication," as its working is less cumbersome. But when multipliers reach three or four figures the partial product method becomes in its turn unwieldy. Yet if we intend to teach our children ultimately to use practice for such cases, we may for a little time suffer the awkwardness of the partial product method and so avoid teaching the long-multiplication method.

In the next section it may be worth while to outline steps by which practice can be taught as a short method of multiplication, as it is fatally easy to "demonstrate" this instead of leaving scope for our pupils' ingenuity as discoverers.

Practice.

1. Set example of type containing £1 and one sub-multiple of £1, awkward for ordinary multiplication.

e.g. Find cost of 342 chairs at £1 3*s.* 4*d.* each.

Take on blackboard different methods adopted by children.

These usually include—

(*a*) Ordinary multiplication of money,
(*b*) $342 \times £1\frac{1}{6} = £342 + £57 = £399$.
(*c*) $342 \times £\frac{7}{6} = £57 \times 7$ (cancelling) $= £399$.

With above example (*a*) is obviously the longest method and usually with hard multipliers children see that (*b*) is shorter than (*c*). It is wise, then, to explain to children that we are to learn a short way of doing multiplication of any sum of money, based on (*b*), and ask them to try next questions on (*b*) plan.

2. Set examples of same type, always containing £1 and a simple sub-multiple of £1.

e.g. Find cost of 365 tons of coal at £1 2*s.* 6*d.* or £1 1*s.* 8*d.* each.

Children's attempts can lead to setting down on board a form ready for extension to harder examples.

e.g.	£365	= price of 365 tons at		£1
	£45 12*s*. 6*d*.	,,	,,	£$\frac{1}{8}$
	£410 12*s*. 6*d*. =	,,	,,	£1 2*s*. 6*d*.

This stage should be practised until children are sure of the fundamental idea, as the remaining steps can then easily be taken by our pupils for themselves. Questions of the following types should be set in succession, each type being mastered before further difficulties are introduced.

3. Set examples involving £1 and 2 simple sub-multiples of £1.

e.g. £1 9*s*. = £1 + £$\frac{1}{4}$ + £$\frac{1}{5}$.

4. Set examples involving £1 and simple sub-multiples of sub-multiples of £1.

e.g. £1 7*s*. 6*d*. = £1 + £$\frac{1}{4}$ + $\frac{1}{2}$ of £$\frac{1}{4}$.
or £1 16*s*. 3*d*. = £1 + 10*s*. + 5*s*. + 1*s*. 3*d*.

5. Set examples involving more than £1 and all previous difficulties.

e.g. £4 16*s*. 3*d*.

6. Set similar examples, increasing in difficulty, but postpone cases where fractional remainders occur until decimalisation of money can be used.

7. Cases where we multiply by a mixed number are also best postponed until decimalisation can be used.

e.g. Find cost of 365$\frac{3}{8}$ quarters at £4 3*s*. 6*d*. per quarter.

£365·375	cost at	£1	each.
£1461·500	,,	£4	,,
£36·5375	,,	2*s*.	,,
£18·26875	,,	1*s*.	,,
9·13437	,,	6*d*.	,,

£1525·441, neglecting unnecessary figures.
Result = £1525 8*s*. 10*d*., to nearest penny.

In above examples we have not made a very full statement, because the method does not serve its true purpose until full statements become unnecessary. But at first it is wise to insist on more detailed explanation, provided children understand that the method is short though the explanation be long, and find that the reward of good work is permission to omit a full explanation. Long statements and endless questions about the cost of "articles" have blinded us to the real value of the practice method as used by business men.

Compound Division.

I. Partition, *i. e.* division of a concrete quantity by an abstract number.

Division of money is usually found difficult because of the complexity of the operation; in early stages much drillwork is required and at later stages children should be encouraged to use decimals as a saving of time. It need not, however, long detain us here. We shall confine ourselves to a few troublesome examples, illustrating the treatment of remainders, division by factors, and a less familiar method of setting down long division, applicable to all weights and measures, and said to give more accurate results than the usual method. It is certainly less cumbersome and unwieldy. It need scarcely be repeated that the difficulty of fractional remainders must be postponed until children have grasped fractional notation. This delay is not serious, as small fractions of a penny are of little importance in actual experience.

Examples—

(*a*) 6 men receive £43 7*s*. 8*d*. If all share alike, what does each man have?

$$\begin{array}{r|rrr} & \pounds & s. & d. \\ 6 & 43 & 7 & 8 \\ \hline & 7 & 4 & 7 \end{array}$$

1. £43 ÷ 6 = £7 each and £1 over.
2. £1 + 7*s*. = 27*s*.: 27*s*. ÷ 6 = 4*s*. each and 3*s*. over.
3. 3*s*. + 8*d*. = 44*d*.: 44*d*. ÷ 6 = 7*d*. each and 2*d*. over.

Result is either, "Each man has £7 4*s*. 7*d*. and 2*d*. is left over,"

or, "Each man has £7 4*s*. $7\frac{1}{3}d.$, dividing 2*d*. left over equally amongst the 6 men and so giving each $\frac{2}{6}d.$ or $\frac{1}{3}d.$

(*b*) 42 men £43 7*s*. 8*d*. If all share alike, what does each man have?

	£	s.	d.	
6	43	7	8	Put 42 men into 6 groups, each of 7 men.
7	7	4	$7\frac{1}{3}$	for each group of 7 men. (See Ex. (*a*).)
	1	0	$7\frac{19}{21}$	for each man.

1. $£7 \div 7 = £1$ each.
2. $4s. \div 7 = 0s.$ each and $4s.$ over.
3. $55d. \div 7 = 7\frac{6}{7}d.$ each.
   and $\frac{1}{3}d. \div 7 = \frac{1}{21}d.$ each.
   $\therefore\ 55\frac{1}{3}d. \div 7 = 7\frac{6}{7}d. + \frac{1}{21}d.$ each $=$ $7\frac{19}{21}d.$ each.

Another treatment of the remainder would be—

1. Each group of 7 men has £7 4*s*. 7*d*. and 2*d*. remains over.
2. Each man, therefore, has £1 0*s*. 7*d*. and 6*d*. remains over for each group.
3. 6 groups were made, $\therefore$ total remainder $= 2d. + 6 \times 6d. = 3s.\ 2d.$

In this case result is, "Each man has £1 0*s*. 7*d*. and 3*s*. 2*d*. is left over.

(*c*) Divide £343 13*s*. $9\frac{1}{2}d.$ equally amongst 102 men.

	£	s.	d.
102	343	13	$9\frac{1}{2}$
	3	7	4
	343	753	$477\frac{1}{2}$
	306	714	408
	37	39	$69\frac{1}{2}$

1. $£343 \div 102 = 3$
   Underneath: $3 \times 102 = 306.$
   Subtract. £37 over.
2. $£37 + 13s. = 753s.$
   $753s \div 102 = 7s.$ each.
   $7 \times 102 = 714.$
   Subtract. 39*s*. over.
3. $39s. + 9\frac{1}{2}d. = 477\frac{1}{2}d.$
   $477\frac{1}{2}d. \div 102 = 4d.$ each.
   $4 \times 102 = 408.$
   Subtract. $69\frac{1}{2}d.$ over.

Result is either "Give £3 7*s*. 4*d*. to each man and $69\frac{1}{2}d.$ or 5*s*. $9\frac{1}{2}d.$ is left over," or—

"Give £3 7*s*. $4\frac{139}{204}$*d*. to each man," for each man gets $\frac{69}{102}$*d*. from 69*d*. and $\frac{1}{204}$*d*. from $\frac{1}{2}$*d*. ∴ each can have $\frac{69}{102} + \frac{1}{204}$*d*. from $69\frac{1}{2}$*d*.

In this case the first result certainly seems more sensible!

It may be well to illustrate this method of setting down by taking examples from other compound rules.

(i) A piece of land 2 miles 7 fur. 13 chains 5 yards long has to be divided into 17 equal parts; find the length of each.

		8	10	22	
	miles	fur.	chs.	yds.	
17	2	7	13	5	
	0	1	4	$6\frac{13}{17}$	is length of each part.
		23	73	115	
		17	68	102	
		6	5	13	

(ii) A quantity of coal weighing 19 tons 17 cwts. 3 qrs. 21 lb. has to be divided equally amongst 217 old women. How much does each receive?

		20	4	28	
	tons	cwts.	qrs.	lb.	
217	19	17	3	21	
	0	1	3	$9\frac{84}{217}$	
		397	723	1461	$(20 \times 72 + 21)$
		217	651	576	(8×72)
		180	72	2037	
				1953	
				84	

In such a case an approximate result for the remainder is the most sensible answer. 84 lb. left over is a waste of almost a bagful; to weigh out $\frac{84}{217}$ lb. or $\frac{12}{31}$ lb. is ridiculous. Obviously each woman can have roughly $\frac{1}{3}$lb. extra; in actual weighing one would simply give good measure for the 9 lb.!

II. Quotition, *i.e.* division of one concrete quantity by another concrete quantity.

e.g. I have to pay men 27*s.* 6*d.* as a weekly wage. I possess £38 12*s.* How many men can I employ?

At a later stage many examples of this type are best worked by fractions. At this stage they can be done only by reduction, children quickly seeing from simple examples that both quantities must be measured in the same unit, sixpences or pence in the above example. This example also illustrates the remainder difficulty, which of course, in early stages, should be avoided.

Measure both quantities in sixpences.

£38 12*s.* = 1544 sixpences:
27*s.* 6*d.* = 55 sixpences.

We now have to divide 1544 sixpences by 55 sixpences.

$$55 \left\{ \begin{array}{r|l} 5 & 1\,5\,4\,4 \\ \hline 11 & 3\,0\,8\ ,,\ 4 \\ \hline & \ \ 2\,8\ ,,\ 0 \end{array} \right\} 5 \times 0 + 4 = 4 \text{ over.}$$

Result is: "Engage 28 men and have 4 sixpences or 2*s.* over," since to engage $28\frac{4}{55}$ men is impossible.

CHAPTER VII

ENGLISH WEIGHTS AND MEASURES

THE fundamental methods of manipulation outlined in Chapter VI. for money are applied also to weight, capacity, length, time, area and volume. If children in each case have clear conceptions of the units in question and of their relations with each other, problems involving addition and the other fundamental operations can be tackled by our pupils without assistance from the teacher unless in exceptional cases. Each new table learnt gives further opportunities for practising the manipulative methods commonly called the "compound rules"; it is not a question of fresh methods each time. After a few introductory remarks applicable to all these tables, we propose, then, merely to give some miscellaneous information about each in turn, if possible treating in more detail any points likely to present difficulty.

Introductory.

1. It is wise to teach the units of measurement in groups, instead of teaching a whole table at once, as in this way we more easily connect Arithmetic with our pupils' life experience.

e. g. Group hours, minutes and seconds round use of clock, but days, months and years round the study of the earth's motion. Group pounds and ounces round shopping, but tons and hundredweights round selling of, say, coal, and hundredweights and stones round selling of potatoes. Group gallons, quarts and pints round selling of milk, but quarters, bushels, pecks round selling of grain.

2. It being most important to give clear conceptions of the units involved, these points should be noted—

(*a*) Build on children's real experience, if necessary giving them new experience.
(*b*) Connect the learning of the tables with the practical use of the units.
(*c*) As far as possible, let children find out for themselves the connections between the units.

These points will be illustrated in what follows.

3. The simpler units of each class, *e. g.* lb. and oz., qt. and pt., hr. and min., yd., ft. and in., sq. ft. and sq. in., are best introduced gradually in lower standards with abundance of practical work; then the tables can be fully developed and formal problems tackled in Standard IV. or thereabouts.

Length.

I. Tables.

Cloth merchants and carpenters require lower part of table, builders the middle part, agriculturists the middle and upper parts. No one person requires the whole table at once.

(*a*) Inch : foot : yard.

Inch.—Latin *uncia*, a thumb-joint. Cp. with length of thumb-joints.

Foot.—Longer than average foot, which in Britain is probably about $10\frac{1}{4}$ inches. At first a varying length.

Yard.—Also a varying length at first. It is said that the length of Edward III's arm was taken as the standard yard. Cp. common ways of measuring a yard of cloth or ribbon, such as distance from chin to finger-tips of outstretched arm, or 8 times the distance from tip to third joint of a woman's curved middle finger.

In connection with fractions, $\frac{1}{2}$'s, $\frac{1}{4}$'s, and $\frac{1}{8}$'s of inches and yards are important. Children should learn how much material to buy for a given inexact measurement: *e. g.* for 23 in. buy $\frac{3}{4}$ yd. or for 40 in. buy $1\frac{1}{8}$ or $1\frac{1}{4}$ yd. as may be possible. Connect these with shopping.

(*b*) Mile : furlong : yard.

Mile.—Latin *mille passus*, 1000 paces. A pace was a double step and, for a soldier, roughly, $1\frac{3}{4}$ yds. Cp. children's own paces and measurement by counting paces.

So 1000 paces can be calculated as approximately $1000 \times 1\frac{3}{4}$ yards or 1750 yards.

Actually 1 mile = 1760 yards, with this advantage that it is easily subdivided into $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$ mile. Children can calculate length of each in yards and time required to walk each, noting time they take to walk a mile.

Furlong.—Teach that in olden days fields were long, and furrows were often about $\frac{1}{8}$ mile long. Hence $\frac{1}{8}$ mile is called a *furrow-length* or *furlong.*

It follows that there are 8 furlongs in 1 mile and 220 yards in 1 furlong.

(*c*) Chain : pole : yard.

Chain.—Used by surveyors and equal to 22 yards. It is length of a cricket pitch. (Surveyors divide it into 100 *Links*, but link is of little general importance.)

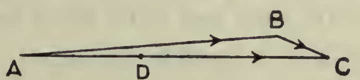
Pole or *Perch.*—A measure which varies in length in different places. Its standard value is $\frac{1}{4}$ chain. Children can see that it is thus $5\frac{1}{2}$ yards, and it can be compared with size of classroom or garden.

Let children find out how many chains in a furlong, by dividing 220 yards by 22 yards; also, how many poles in a furlong, there being 4 poles in a chain. Then complete table can be formulated.

II. Suggestions for Practical Work.

The more familiar types of measurement lessons need not detain us. We are all aware of the work that can be done by little children with foot-rules and yard-measures. Perhaps some may forget the value of allowing children to estimate or "guess" a measurement before taking it, a practice of value in dealing with most forms of measurement and very attractive to children, who love "guessing" games.

When we come to deal with the measurement of greater lengths, the question of apparatus is sometimes troublesome. Often tape measures can be borrowed from the needlework cupboard, and many schools possess one or more surveyor's tapes. Where these cannot be had, all that is necessary is a supply of string and paper. Six to ten paper strips a yard long, subdivided into feet or inches, or $\frac{1}{4}$'s or $\frac{1}{8}$'s of a yard as desired, and pinned to walls of room or against desks, will keep a large class busy without confusion. The long measurements are taken by string and the string is then measured by these paper strips. Care must be taken with regard to measurements of greater length than the string or tape used, as it is easy to measure a crooked line instead of a straight one, if no wall or street guides us; *e. g.* we measure AB and BC and say that the sum is the length of AC.



If AC is too long for measurement by one application of string or tape, we must apply the surveyor's art of "sighting"; *i. e.* we must set up a pole at, say, D, in such a way that when we look from A, C is hidden. This means that A, D and C are in one straight line. Then we measure AD and DC, the sum of which lengths is equal to AC. Obviously two children can mark A and C, while a third, D, moves between them until he is in the line AC.

Good practice in this type of work arises in connection with the making of plans or maps in Geography. We suggest here two lessons on a less familiar plan, hoping that they may set some people to more original developments of their practical Arithmetic. The aim in each case was to provide practical work along with calculation as we have them in life outside school, and to make the class realise the advantages of working in groups or as a class towards some goal a single individual would find it difficult to reach.

(a) *On buying Material for Articles requiring Approximate Measurement.*

One or more lessons on $\frac{1}{4}$'s or $\frac{1}{8}$'s of a yard, and on buying cloth and ribbon in that way instead of in inches, had been taken beforehand.

Class was subdivided into groups of three or four, and a number of problems equal to the number of groups, with perhaps six extra ones, was sufficient. The problems were of this type and referred to articles at hand for children's measurement.

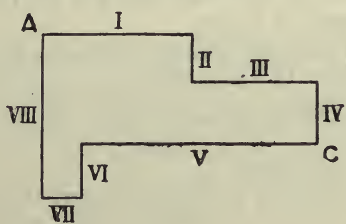
> "New lace has to be provided for the white tray-cloth. It is put on plain, with fulness only at corners. How much lace must you buy, if you allow 3 in. extra at each corner for gatherings and 1 in. for joining? Find the cost at $5\frac{1}{2}$*d.* a yard."
>
> "How much serge must we buy to make a curtain for the classroom door, allowing 2 in. at top for a hem and 3 in. at foot, curtain to be $\frac{1}{2}$ in. off the floor? Find cost at 1*s.* 11*d.* a yard."

Each problem was placed in an envelope with an indicating sign outside for teacher's use (*e.g.* "tray-cloth," "door"). Inside the envelope in general should be placed also a tiny slip with the answer. Children must show correct working to teacher before obtaining another envelope, but it saves waste of teacher's energy if children have some check on their work without troubling her except in cases of difficulty. If undue noise is guarded against, a noisy group, if need be, being sent back to written work, the teacher will be amazed at the amount of work done and the business-like activity and ingenuity of the children. For a first lesson of this type, the teacher will have the trouble of preparing answers, but the work need never again be done in its entirety.

(b) *On Measurement of Playground to illustrate Length of Furlongs, Chains and Poles.*

This question was proposed to the class: "How far would you walk in going round the playground, keeping close to the walls? How many times could you run round it in 10 minutes, if you can run at 6 miles an hour?"

Class, having previously looked at playground, then drew rough diagrams of it. Their work was checked and elaborated by a final sketch on blackboard of the form of the playground. Children decided how many measurements must be taken, and that the measurements were to be in yards and feet, correct to nearest foot. The class was divided into groups, each to be responsible for one measurement; it is even better if two groups do the same measurement and check each other's work. Yard paper strips marked in feet were pinned up round room and each group received some string. They went out, took their measurements, came back, measured string by paper strips and entered the results on blackboard, *e. g.*:—



I = 20 yds. 1 ft.	V = 31 yds.
II = 6 yds. 1 ft.	VI = 7 yds.
III = 16 yds.	VII = 5 yds. 1 ft.
IV = 8 yds.	VIII = 21 yds. 1 ft.

It should be noted that I + III = V + VII and II + IV + VI = VIII.

$\therefore$ total distance = 2 (VIII + I + III).
= 2 (57 yds. 2 ft.).
= 115 yds. approximately.
= approximately $\frac{1}{2}$ furlong.
or 5 chains.
or 20 poles.

The working out of question asked should be done by the class. Other exercises can be done in this and the subsequent lesson.

e. g. Which wall is most nearly a pole, **3** poles, a chain, $1\frac{1}{2}$ chains?

If you take **22** paces in walking straight from wall VIII to wall IV, what is the length of your pace?

Find your rate of running by finding how long you take to run twice round the playground.

Draw a plan of the playground and find from it the distance straight from A to C. Check your result by measurement.

III. Note on Reduction of Yards to Poles.

This is of little value in itself, but when of necessity taken, it may present difficulty.

e.g. How many poles and yards are there in **416** yards?

We have to find **416** yards $\div$ $5\frac{1}{2}$ yards, because $5\frac{1}{2}$ yards $=$ **1** pole.

We must measure both in half-yards, before dividing.

$5\frac{1}{2}$ yards $=$ **11** half-yards.

$$\begin{array}{r|l} & 416 \text{ yards} \\ & \quad\; 2 \\ \hline 11 & 832 \text{ half-yards} \\ \hline & \;\; 75 \text{ groups of } 11 \text{ half-yards and } 7 \text{ half-yards over.} \end{array}$$

i.e. **75** poles ,, $3\frac{1}{2}$ yards.

Area and Volume.

We refer the reader to Part II, Chapter XVI, on Areas and Volumes, noting merely that the tables must be built on the tables of length, the most important units of area being square inches, square feet, square yards, acres and square miles, and of volume, cubic inches, cubic feet and cubic yards.

Capacity.

I. Meaning of Capacity.

The simplest method of selling some goods is to measure them by putting them into some vessel which is known to contain a given amount; in other words, we measure them by the quantity standard vessels can hold, or by " capacity." Children often have the vaguest notions of the meaning of the word "capacity"; "measure of captivity" seemed to a child equally sensible. The underlying idea is so simple that it is worth while to explain the word and the method of measurement based thereon. Children should discuss which goods are best sold in this way; the reasons why some things are always sold by capacity (*e. g.* milk, vinegar), some always sold by weight (*e. g.* biscuits, meat) and some in places by capacity and in places by weight (*e. g.* strawberries, currants, potatoes, grain); and the advantages and disadvantages of capacity as compared with weight. See succeeding section on weight.

II. Tables.

(*a*) Pint, ½ pint, ¼ pint.

Cp. with tumblers, breakfast- and tea-cups, and refer to milkman.

A standard gill is ¼ pint, but in the north of England a gill is ½ pint; which causes occasional confusion.

(*b*) Gallon, quart, pint.

Children should use measuring vessels and find out for themselves that

1 quart = 2 pints.
and 1 gallon = 4 quarts.

(*c*) Quarter, bushel, peck, gallon.

Here direct acquaintance with units is difficult, as they are used mainly for selling grain and large quantities of some vegetables. Usually we must state the relationships—

2 gallons = 1 peck,
4 pecks = 1 bushel,
8 bushels = 1 quarter,

and illustrate diagramatically, as suggested in the following note on practical lessons. Fortunately children will not require the same degree of familiarity with these in their life after school.

(*d*) Learning capacity table.

Capacity table is the one table of weights and measures which must be learnt by heart and said through when a given relation is required, because of the confusion of 2's and 4's. At the same time the relations up to quarts usually become so familiar that formal repetition of the table is unnecessary for measurement of small quantities. This table is most useful for giving mental problems involving application of relations between halves, quarters and eighths.

III. Suggestions for Practical Work.

The necessary apparatus includes at least one set of standard vessels, up to quarts, and as good a collection of miscellaneous jugs, basins, cups, bottles, kettles, pails or even barrels as can be achieved by the joint labours of teacher and class. Pailfuls of water can be used with remarkably little mess for purposes of measurement, but some may prefer to have a supply of fine sand. If a school lacks standard vessels, the teacher can usually find somewhere a jug, a basin, a tumbler and a cup of suitable size to serve in their place. We suggest these types of practical lessons, as they give good results in making units more familiar and in showing the relationship between capacity and weight.

(a) *On Measurement by Capacity, involving use of Pints, Quarts and Gallons.*

Preliminary work can be done in connection with learning of units, by letting children estimate how much the vessels familiar to them (*e. g.* jugs or kettles) hold and then checking their results by measurement.

The method would follow the lines of the lesson suggested on p. 98. The ordinary supply of standard vessels will probably have to be augmented by using jugs or cups already measured by class and having their capacity marked on them by, say, a gummed label.

The problems set would be of this type: "By guessing the capacity of this bottle, find the probable cost of filling 10 such bottles with vinegar at 8*d*. a pint. Then check your result by measuring the capacity of the bottle and calculating the cost accurately."

(b) *On Diagrammatic Representation of the Units of Capacity.*

Children have each a page of ordinary squared paper, and are asked to devise a way of illustrating on it the units from a pint to a quarter. The better part of the class will probably make good suggestions and be able to devise for themselves good diagrams. The slower children usually require more help and must have their work subdivided into stages.

e.g. A pint can be represented by shading 1 square.
A quart, alongside the pint, would require 2 squares.
A gallon, being 4 quarts, 8 squares; a peck, 16 squares, a bushel 64 squares (= 6·4″).
A quarter would have to be represented differently, as paper would not be 8 × 6·4″ long, say by 8 columns each 6·4″ long.
A variety of colours and an explanation of the units each colour refers to, is of use with slower children.

Such a lesson gives children some idea of the comparatively great size of a bushel and a quarter. Problems on it will keep the quick children busy while their slower comrades finish what is for them very useful work, not to be left half done.

(c) *On Solution of a Problem involving Weight of a Known Quantity of Water.*

This problem was set. " A boy had to be able to carry a watering-can full of water. It held two gallons and, when empty, weighed 2 lb. Find what weight he had to carry."

Children soon realised the difficulty and decided that the weight of a gallon of water must be found. A gallon vessel was missing, and in any case would have been rather large for the balance provided. They decided to try the quart measure, but found it difficult to weigh when full of water. They finally used a pint of water, weighed inside the quart vessel, and were quick to see that the weight of the quart vessel must be subtracted. Having found the weight of a pint of water to be about $1\frac{1}{4}$ lb. they were left to finish the problem set and a good part of the class got a solution for themselves. Other problems were then set involving the new knowledge learnt, *e.g.* on weight involved in carrying water-bottles.

Older children should be asked if it would make any difference if the can were filled with milk or mercury or oil, and so led to appreciate the significance of density or specific gravity; but such work is beyond the limits of the brain of the average Standard IV child.

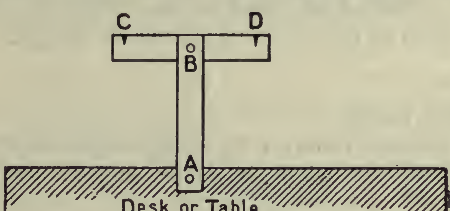
Weight.

I. Meaning of Weight.

The idea of selling goods by weight is less simple than the idea of selling by capacity, for in this case we are drawing on the hidden forces of the universe to aid us; familiarity blinds us to the wonder of this. But although any full explanation is beyond the grasp of a child, as it is, in truth, beyond the grasp of any one of us, yet even a child may feel something of the awe which best becomes man in face of such mysteries. Some children, too, need the comfort of knowing of the existence of gravitational force. It is but one more strange fact about Mother Earth. At first we thought she was flat; but we were told and convinced that she was round and felt thankful that we at least lived on the upper side of her. At first we thought her motionless; but we were told and convinced that she whirled through space at a dizzying speed and we became afraid of falling off into a horror of empty air. Surely, then, we might have been told of those strange invisible arms with which Mother Earth draws all things to herself, by which she keeps us from falling away into space. Every time a child jumps off the earth, he feels the pull of those arms drawing him back; nothing pulls him back when he practises a long, instead of a high, jump. This strange pull acts on each pan of the balance as it hangs in mid-air, unless the other pan is acted on by a great enough pull to hold its neighbour up, just as with two boys on a see-saw. When we measure goods by weight we compare the pull of the earth on the goods to be sold with the pull of the earth on some known quantity of metal, and so measure out a given quantity; for the greater the thing or person, the stronger is the pull of Mother Earth's invisible arms. Of course, this is a very inadequate illustration; it tells nothing of the other side of things, of how we attract the Earth as she attracts us, of the effect of distance on gravitational force; it ignores the spring balance; it makes no attempt to distinguish between mass and weight, and might even tend to a confusion of mass and volume. But it should do a little towards providing the right atmosphere, by carrying us out of the hot-house of mechanical calculations into the wide spaces of the breezy heavens. Provided it is not given as an adequate explanation, but simply as a hint at truth of which later some of them may know more, it must do good and not harm.

II. Apparatus.

The provision of sufficient apparatus to allow of all members of a class using a balance, is a problem. To tiny children, rough scales made from string and the lids of round tins give much pleasure and a certain intuitive knowledge of weighing. A simple arrangement of supports for these pans could be fastened to the backs of children's desks at a very small cost; the making might be done by older boys. All that is necessary is a movable framework consisting of two strips of wood, jointed at A



and B, with notches at C and D for hanging string over. When not in use, it could be folded along the back of the desk, CD first rotating round B, until lying along BA; the whole then rotating about A until lying along the back of desk. Other more temporary supports are devised by many teachers of little children.

Then some of our older pupils, in learning science, will have practice in the use of very accurate balances, but most girls and a good many boys never reach this stage. The intermediate stage, reached in Standards II, III and IV, requires that children shall have an opportunity of using balances in common use in the workaday world. One and occasionally two such are all the school possesses. No child can weigh for another. What are we to do? For one thing, in some schools use can be made of home. Perhaps occasionally from a home near school an extra balance can be borrowed; more often children will be allowed to practise weighing at home if given a series of definite problems requiring weighing and asked to hand in the answers: *e.g.* "How many pennies must you take to balance a 4-oz. weight?" or "Find the weight a boy lifts in carrying to the cupboard 20 books like your Arithmetic book." "Bring to school in a bag ½ lb. of sand."

If practice in weighing can be given only in school with not more than two balances, the best plan seems to be to have a series of questions involving weighing, to which answer is known, and withdraw from lessons that consist mainly of individual written work groups of two or three children, allowing each group, say, five or ten minutes at the balance. Some use can be made of intervals and the dinner hour, but children's great desire to weigh should somehow be satisfied and used to lend interest to the less attractive parts of the series of lessons on weight.

III. Tables.

(*a*) Pound; ounce.

Let children discover by using weights and packets of sand that there are 16 oz. in 1 lb. Teach use of ½ and ¼ lb. weights.

(*b*) Hundredweight; quarter; stone; pound.

The hundredweight, *i.e.* 100-wt., C-wt., or cwt., was originally 100 lb., but a varying number of extra pounds was added on for "discount";

cp. a baker's dozen. In the reign of Queen Elizabeth this extra amount was fixed by law as 12 lb.

$\therefore$ 1 cwt. = (100 + 12) lb. =	112 lb.	Cp. sack of coal.
and $\frac{1}{2}$ cwt. =	56 lb.	Cp. average weight of Standard IV child.
$\therefore$ $\frac{1}{4}$ cwt. or a *quarter* =	28 lb.	Cp. average weight of 2-year-old child.
and $\frac{1}{8}$ cwt. or a stone =	14 lb.	Note use in shopping; also that stone is for some purposes 8 or 12 or even 16 lb.

From this children can of course build up table from pounds to hundredweights as usually given. But above plan helps to give children clear conceptions of unfamiliar units and gives a reason for the table being as it is. It should be connected with the use of the school weighing machine and their own weights.

(*c*) Ton; cwt.

Teach that 1 ton = 20 cwt. Perhaps children have counted the number of sacks of coal in a 1- or 2-ton load, or noted how many tons coal-trucks, seen on the railway, are said to carry, this being marked on the side of each truck.

It is useful too if children have an opportunity of seeing carts and heavy loads weighed by the steelyard in the ground. Again the explanation is too difficult for them, but one day some of them will understand.

Time.

This table depends not on man's arrangement, but on the motion of the heavenly bodies. Here, for fundamental units, man accepts Nature's ruling; in measuring other kinds of quantities he fixes his own units. We shall take the units in groups, throwing out miscellaneous suggestions which may be useful to some teachers in their attempts to give their pupils clear conceptions of the units and of their relations with each other. A good piece of preparation for oneself is to meditate on the mystery of man's appreciation of the lapse of time, on what we mean by "time" and on the different methods by which it has been measured, *e.g.* hour-glass and sundial.

(1) *Hour; minute; second.*

Children can sit still 1 minute, 1 second, and describe what might happen in any 1 hour of the day. They should make clock-faces for themselves and practise reading the clock in different ways. A set of railway time-tables, or copies of a page of a time-table, afford excellent practice in the two ways of stating time (*e.g.* 3.43 or 17 minutes to 4) and a variety of problems of a type obviously useful to children. We should teach them how to count from one time to another, say, to find the length of a journey or the best train; we naturally introduce the distinction between 9 o'clock before noon, or 9 o'clock after noon, which, instead of writing as 9 b.n. or 9 a.n., we write as 9 a.m. or 9 p.m., because the Romans used the words "ante meridiem" for "before noon" and "post meridiem" for "after noon." In our own minds we should distinguish clearly between hour, minute and second *spaces* on the clock and the actual times which correspond. Children can be set thinking for themselves if brought to discuss whether they can *see* time!

(2) *Year; day; hour.*

Our clocks, however, must all keep time with the great clock of the world, that is, the Sun. Reference can be made to the use of Greenwich time, and in upper standards this work may be correlated with Geography by taking the difference in time at different places on the Earth's surface. For the present, at any rate, we shall also have to deal with the effects of the Daylight Saving Act.

(*a*) Earth turns completely round once in 24 hours, which we call a complete *day*. But children find this use of "day" ambiguous, as they think of "day" in contrast to "night." Children should note when a complete day begins and ends, *e.g.* when the date changes, and describe what happens in a day as the hour-hand of a clock makes two complete revolutions.

Children can dramatise the motion of the earth with regard to the sun, which produces day and night, and quarters of a complete day should be taken in preparation for the discussion of the leap-year difficulty. Some schools will be fortunate enough to have apparatus for illustrating the earth's motion, but it is not absolutely necessary, as children's dramatic instincts can be appealed to.

(*b*) If some boy is asked to spin a top and the class observes exactly what happens, we shall get a simple line of approach to the twofold motion of the Earth. The top rotates on its own axis, but also moves over the floor. So the Earth moves, like a wonderful top, not resting on a floor, but held in its place by the invisible arms of the Sun (cp. note on weight, p. 104); not moving in an unsteady fashion, but revolving with perfect balance in a beautiful curve, very like a circle. Again children can dramatise, can note that the earth moves round the sun in this curve once a year, the seasons of Winter, Spring, Summer and Autumn thus being given us in turn. They can discuss when 1917 began and when it ends; they can settle for themselves by whose decision it is that each year is just as long as it is! The exact length of a year is best treated in three stages, corresponding roughly to Standards III., IV. and VI.

(i) When the idea of the rotation of the earth is new, we should teach that in making this "circle" round the sun the earth turns round about 365 times.

$$\therefore 1 \text{ year} = 365 \text{ days.}$$

Children should be able to explain why there are 365 days in a year.

(ii) Next time, the leap-year difficulty must be discussed. Does the earth have a lazy year once in every four? This leads to the distinction between a calendar year and a sun or solar year. We have to make our calendars keep time with the sun and the earth, and every year the earth rotates about $365\frac{1}{4}$ times in moving round the sun.

$$\therefore 1 \text{ solar year} = 365\tfrac{1}{4} \text{ days.}$$

e.g.

Date.	Sun year.	Calendar year.	At end of year calendar is
1913 1914 1915 1916	$365\frac{1}{4}$ days $365\frac{1}{4}$,, $365\frac{1}{4}$,, $365\frac{1}{4}$,,	365 days 365 ,, 365 ,, 366 ,,	$\frac{1}{4}$ day late $\frac{1}{2}$,, ,, $\frac{3}{4}$,, ,, 0 ,, ,,
	1461 days in 4 sun-years.	1461 days in 4 calendar years.	

We should also teach the rule for finding leap-years; the first would be 4 A.D., 8 A.D., 12, 16, and so on. Therefore we take as leap-years those dates divisible by 4. But since 100 is always divisible by 4, any number of hundreds, *e.g.* 1800 or 1900, is divisible by 4. Hence we require to test only last two figures: *e.g.* 1834 is not a leap year, as 34 is not divisible by 4, but 1916 is a leap year, because 16 is divisible by 4.

(iii) But at a later stage we may teach that it is very difficult for the calendar to keep exact time with the earth, as the earth rotates not exactly $365\frac{1}{4}$ times in one year, but a little less. Thus, if we put in a leap-year every four years, the calendar would go ahead of the solar system; therefore three times out of every four hundred years, a leap-year is omitted at the beginnings of centuries.

e.g. 1700, 1800 and 1900 are not leap-years; 1600 and 2000 are.

(3) *Year; week; day.*

The only thing necessary is to let children discover how many weeks there are in a year, why their birthdays and Christmas Day move one day forward each year, and how they change after leap-years.

(4) *Year; month; day.*

The development of our present system of months has been a long and complicated one, and the reader who desires to know more is referred to any good encyclopædia. Meantime we confine ourselves to a simple outline suitable for children, and helpful because it does give a certain historical background.

We first note that the word "month" is connected with the word "moon," and often means simply four weeks. But for convenience' sake the year as a whole came to be divided into twelve periods, called months. Suggest that children try to divide the year in this way and see if the calendar they make is like the present one.

(i) 365 days ÷ 12 allows 30 days to each month and 5 days over.

Let children suggest plans for dealing with 5 extra days. Various plans will be suggested. One actually taken was to add 1 day to each alternate month. But this added 6 extra days. However, the idea of long and short months coming alternately was satisfactory and the necessary sixth day was taken from February. See Columns I. and II. in the following table.

(ii) Two of the greatest Roman rulers who ever lived were Julius Cæsar and Augustus Cæsar. Let children find the months named in their honour, and note that Julius was honoured with 31 days, while Augustus had only 30. Those who favoured Augustus said that his month also must have 31 days. This new extra day had to come from somewhere and was taken from poor February. See Column III.

(iii) But this altered the balance of the months, giving 3 long months on end—July, August and September. Children will probably suggest making September and November short, October and December long, to recover some of the balance. See Column IV.

(iv) Naturally in a leap-year the extra day goes to February. See Column V.

This is an historical outline of how our calendar came to be the complicated affair it is, and children will quickly see the necessity of learning the old rhyme—

> "30 days hath September,
> April, June, and November:
> All the rest have 31,
> Excepting February alone,
> Which hath but 28 days clear
> And 29 in each leap year."

Children should use the rhyme until they know the number of days in any month.

	I.	II.	III.	IV.	V.
Jan.	30 days + 1	31	31	31	31
Feb.	30 ,, − 1	29*	28*	28	29
Mar.	30 ,, + 1	31	31	31	31
April	30 ,,	30	30	30	30
May	30 ,, + 1	31	31	31	31
June	30 ,,	30	30	30	30
July	30 ,, + 1	31	31	31	31
Aug.	30 ,,	30*	31*	31	31
Sept.	30 ,, + 1	31	31	30	30
Oct.	30 ,,	30	30	31	31
Nov.	30 ,, + 1	31	31	30	30
Dec.	30 ,,	30	30	31	31
	360 + 6 − 1	365	365	365	366

The only difficulty in the arithmetic of the months is in cultivating ability to count from one date to another and deciding when to count both dates and when only one. This can be subdivided into easy stages—

(*a*) Between two dates in same month: *e.g.* August 17–21. Note that simple subtraction gives correct result if only one day is counted.

(*b*) Between a date and the end of its month: *e.g.* August **17** to end of August, *i.e.* **31–17** days.

(*c*) Between two dates in consecutive months: *e.g.* August **17** to September **8**.

August **17** to end of August = **31–17**, not counting **17**.

∴ Total = **31** − **17** + **8**, counting September **8** = **22** days.

(*d*) Between any two dates in the same year.

(*c*) Between any two dates in any years.

CHAPTER VIII

THE METRIC SYSTEM OF WEIGHTS AND MEASURES

In striking contrast to our complex English system of weights and measures, with its loosely knit interrelations and awkwardness for calculation, is the compact metric system, unified and simple. The one bears the marks of the slow, irregular growth of mankind in measurement of quantity; the other was rapidly created by the wisdom of a well-developed race at a certain time and for a definite purpose.

The French nation at the time of the French Revolution set themselves definitely to develop a system of scientific measurement. In 1790 a decree of the National Assembly at Paris appointed a committee to consider the suitability of adopting, as a scientific unit of length, either the length of a pendulum beating seconds, or a fraction of the length of the equator, or a fraction of the quadrant of a terrestrial meridian. The committee reported in favour of a fraction of a quadrant of the meridian passing through Paris. The Assembly then appointed two commissions, one to measure an arc of this meridian, the other to draw up a system of weights and measures based on this measurement and to decide on suitable nomenclature. In December 1799, the National Assembly fixed the value of the metre and of the kilogramme, by a law based on the reports of these two commissions; in 1801 they made the new system compulsory. It was, however, necessary in 1837 to pass another law forbidding, after January 1840, the use of any other weights and measures, under severe penalties.

Such a novel system of measurement, in spite of its simplicity and usefulness, took time to win its way into the haunts of ordinary life and work. As matters now stand, the metric system is compulsory in practically every country in Europe; the list includes France, Germany, Austria-Hungary, Greece, Italy, Rumania, Serbia, Netherlands, Belgium, Switzerland, Norway and Sweden. It has also been legalised in Egypt, Russia, Turkey, Japan, the United States and Great Britain. In Britain our scientists use it universally; in commerce, because of international relationships, we use it along with our own complex system; but in the affairs of everyday life as they touch the home, the shop and the workshop, we adhere to the English system. So slow, indeed, have we been in our appreciation of the metric system, that although a Decimal Association has been working in its favour for the past sixty-six years, it was only in 1897 that the Weights and Measures Act legalised its use in trade and abolished the penalty for using or having in one's possession a weight or measure of the system.

The main reason against change is that for a time complications would be unavoidable, as people would either gain or lose by the introduction of the system, while many would be unable to measure by the new units. The main reasons for a change are that we should place ourselves in closer contact with other nations, we should greatly simplify commercial relationships and calculations, and we should be able to save at least two hundred hours of a child's school time for work of higher value to the individual and the race. Meantime, in teaching the metric system, we should realise that, besides preparing our pupils for the demands which science or commerce may later make on them, besides giving them a more sympathetic understanding of life in other lands, we are making them ready for a change which is almost certain to come before some of them have reached the end of their days. Already our closer relations with our Allies in the war are hastening the introduction of the new units of measurement.

All that has been said in the introduction to the chapter on English weights and measures applies here also. We must do our utmost to make the units familiar to our pupils, and we must let them discover their relation to our own units. We must also correlate the teaching of the metric system with the teaching of decimals, though, for the purposes of this book, we postpone the treatment of decimals to a later chapter.

Length.

In attempting to devise a scientific unit of length, the commission appointed by the French National Assembly fixed their attention on that great circle on the earth's surface which can be drawn through Paris and the North and South poles. They measured a known arc of this circle as accurately as possible and, acting on the results they obtained, stated that the length of $\frac{1}{10,000,000}$ of a quadrant of this circle was the standard unit of length, to be called a "metre." Later work has, however, proved that the length they called a "metre" does not bear the same relation to the earth's surface that they supposed, so that a metre is now best defined as the length of a standard bar of platinum kept at standard temperature by the Government of France. Its length, when compared with our units, will be found to be about $39\frac{3}{8}$ inches, or rather more than 1 yard. It is used in general where we should use the yard.

Multiples and sub-multiples of the metre were to increase and decrease by 10, so that the calculations involved might be closely akin to the ordinary fundamental operations. Fixed prefixes were selected for the multiples: deka = 10; hecto = 100, kilo = 1000, Greek roots being used. Others were selected for the sub-multiples: deci = $\frac{1}{10}$, centi = $\frac{1}{100}$, milli = $\frac{1}{1000}$, Latin roots being taken. The close connection of this with our decimal notation needs no comment.

The dekametre and hectometre are of less importance, but every child should be able to compare a kilometre with a mile; illustrations of the necessity for this abound in these days of continental warfare. Children can easily find how many yards or furlongs are in a kilometre, by reducing 1000 metres to inches. They find that a kilometre is approximately 5 furlongs or $\frac{5}{8}$ mile, or, assuming that they walk a mile in 20 minutes, a kilometre is the distance walked in $12\frac{1}{2}$ minutes. The application of these results to practical problems involving distances known to the children, is necessary if the new idea of the kilometre is to have sufficient significance for our pupils.

Our pupils should practise estimating shorter distances in metres, decimetres, centimetres and millimetres, and should check their estimations by measurement. If metre rules cannot be had, strips of stiff paper a metre long can be pinned up in classroom (as suggested in Chapter VII, p. 97, for yard measures), while good 12-inch rules usually show graduations for the smaller metric units. When children have noted that a decimetre is, roughly, $\frac{1}{3}$ foot, have proved that 1 inch is approximately equal to 2·54 centimetres by measuring in centimetres a line, say, 5″ long, have applied this new knowledge to problems of interest, for example, to a comparison of the sizes of well-known French and English guns, they have done the minimum amount of work necessary to give the metric system a sound foundation in their minds. Teachers will naturally develop the subject on lines suited to their type of class and their apparatus.

Area.

The units of length once grasped, area should present only one point of real difficulty. Obviously we have as units square metres, square decimetres, centimetres and millimetres, square dekametres, hectometres and kilometres. Obviously too, the table will rise by 10^2 or 100; as there are, for example, 10 decimetres in a metre, in a square metre we have 10 rows of 10 square decimetres, or, altogether, 100 square decimetres.

There is, however, one drawback to this arrangement. If we think of a square dekametre, we can visualise it as a square patch of ground, about 11 yards each way, the area of a good-sized square room. If we think of a square hectometre, we find by reducing it to square metres, and then changing it to square inches, square yards and acres, that it comes, roughly, to $2\frac{1}{2}$ acres, which, as we know, is sufficient ground for several fair-sized houses. Obviously in this part of the table we are dealing with the sizes of pieces of land constantly being measured for building purposes, and we can see the awkwardness of having no unit between the square dekametre and the square hectometre. To overcome this difficulty and obtain an intermediate unit is impossible so long as the metre and its multiples are used as the basis for the units of area.

Hence a new unit is introduced, called an *are*. This is defined as equivalent to 1 square dekametre, that is, to the area of a fair-sized room. The are is then treated as the metre in the linear table: a decare is 10 ares, a unit intermediate between square dekametres and square hectometres; a hectare is 100 ares, equivalent to a square hectometre or, roughly, $2\frac{1}{2}$ acres. Thus the difficulty is overcome and we have a twofold table of area—

100 100 100 100 100 100
sq. Km. sq. Hm. sq. Dm. sq. metre sq. dm. sq. cm. sq. mm.
‖ ‖
10 10
Ha. Da. are

Volume and Capacity.

Volumes greater than a cubic metre being seldom measured, we may take as the units of volume the cubic metre, cubic decimetre, cubic centimetre (usually written c.c.), and cubic millimetre. The relationship in this case is not 10, nor 100, but 1000 or 10^3. For example, in a cubic metre there are 10 square layers, each 1 decimetre thick and of cross section 1 square metre. In each of these layers will be 10 rows of 10 cubic decimetres or 100 cubic dms.; therefore in a cubic metre, with its 10 such layers, will be 10×100 or 1000 cubic decimetres.

But here again, in practice, a difficulty arises. If our pupils make a cube of stiff paper or thin card, 1 centimetre each way, and another cube, 1 decimetre each way, they will see for themselves that for purposes of buying milk, cream, beer or vinegar a cubic centimetre is too small and a cubic decimetre often too large. The difficulty is again overcome by the introduction of a new unit, in this case the *litre.*

A litre is defined as equivalent to 1 cubic decimetre; a decilitre will be $\frac{1}{10}$ cu. dm. or 100 c.c., a new intermediate unit; a centilitre will be $\frac{1}{100}$ cu. dm. or 10 c.c.. a second intermediate unit; a millilitre is equivalent to 1 cubic centimetre. We have, thus, a twofold table for volumes and capacity—

			1000			1000	1000
cu. metre			cu. dm.			c.c.	cu. mm.
‖			‖			‖	
	10	10	10	10	10	10	
Kl.	Hl.	Dl.	litre	dl.	cl.	ml.	

As, like the metre, the litre is in constant use on the Continent, children should spend some time in making its acquaintance. It should be compared with the pint; it measures approximately $1\frac{3}{4}$ pints. The amount of milk to be bought per week by a French family of given size should be worked out. The sizes of tumblers, cups, jugs, basins and bottles should be estimated in the new units and these estimations corrected by measurement.

Weight.

The unit of weight is obtained from the unit of volume in the following way. A cubic centimetre of pure water at standard temperature is weighed, and its weight is taken as the fundamental unit of weight, called a "gram."

Children can use the cubic centimetre cubes made in connection with measurement of volume and try to compare the weight of water such a cube holds with the tiny gram weight. A better plan is to compare with a kilogram weight, the weight of water held by a cube whose sides are one decimetre long, or by a litre vessel. Some such exercise is necessary to establish the relationship between weight and volume.

The subdivisions of a gram are used by scientists and our older boys may use them in connection with accurate weighing for scientific purposes; but for the average child in the average class they need not be emphasised. The weight of real importance in daily life is the kilogram; the children should compare this with our pound weight and discover that it is approximately 2·2 lb. They should also compare the 100-gram weight and the 10-gram weight with our half-pound and ounce weights. If in addition to this they estimate the weights of some familiar objects in metric units, and check their estimates by actual weighing, the metric system for weight should have a sufficiently firm foundation.

For measurement of greater weights, a "tonne," which equals 1000 kg., and a "quintal," which equals $\frac{1}{10}$ tonne, are in common use. An appreciation of these weights is very easy, because a tonne corresponds almost exactly to our ton, and a quintal, therefore, is about 2 cwt. The reasoning is very simple—

$$\begin{aligned} 1 \text{ tonne} &= 1000 \text{ kg.} \\ &= 1000 \times 2{\cdot}2 \text{ lb. approximately.} \\ &= 2200 \text{ lb. approximately.} \\ 1 \text{ ton} &= 2240 \text{ lb.} \end{aligned}$$

CHAPTER IX

VULGAR FRACTIONS

ALTHOUGH the newness of the fractional idea and of fractional notation involves us in a large amount of slow preparation before tackling in detail the manipulation of fractions, yet if this preliminary work be thoroughly done, it is surprising how quickly the greater part of a class afterwards advances. Teachers should not be disappointed because a certain minority requires additional practical work and drill in setting down written work before fractions are mastered; the fundamental principles are so new to children that they need time to sink into the minds of slow learners.

To ensure this gradual development of fundamental ideas, two plans should be adopted. In the lower standards the idea and notation of simple fractions should gradually be introduced in connection with practical work. For instance, halves and quarters of **1**_d._, **1**_s._ and **£1**, and eighths of **1**_s._ and **£1** can be illustrated by drawing and folding of lines, rectangles or circles, and applied to the calculation of prices; halves and quarters of an hour arise in learning to read the clock; halves, quarters and eighths of yards, pounds and quarts in shopping; halves, quarters, eighths and even sixteenths of an inch in connection with measurements for handwork and drawing of plans. Thirds, sixths and twelfths can be introduced from **1**_s._, from **1** ft. and from hexagons in circles; see Part II., Chapter XV. on Geometry. Tenths arise early in connection with measurement, along with the beginnings of decimal notation and of the metric system. From such introductory work in the first three standards, children should learn gradually that the lower figure in a fraction *names* the kind of parts taken or the number of parts into which the whole has been divided, and is called the *naming* figure, Latin *denominator;* while the upper figure gives the *number* of these parts actually taken and is called the *numbering* figure, Latin *numerator*. The Latin names may, of course, be deferred till late in school course. Children should also learn from this early work that the more parts a whole is divided into (*i. e.* the greater the denominator), the smaller will these parts be, and therefore the greater the number of them that must be taken to be equivalent to some simple part of the whole. If the whole be divided into twelfths, say, we shall require 4 of them to be equal to one third of the whole, and so 8 twelfths to be equal to 2 thirds. In other words, the greater the denominator, the greater the numerator in a set of equivalent fractional quantities. To children well grounded it should be obvious that $\frac{1}{7}$ is greater than $\frac{1}{9}$ and so $\frac{4}{7}$ than $\frac{4}{9}$, this although it seems to turn their previous experience of figures upside down! Provided also that children meet fractions first in connection with measurement, they realise that fractions are always fractions of some concrete quantity and so have meaning; many little ones in their day have seen no sense in fractions. When simple addition, subtraction or multiplication problems arise naturally out of this practical work, they need not be avoided, but the formal learning of the rules for manipulation of fractions is best postponed till Standards IV. and V.

In the second place, when such formal work is at last reached, the necessity for a thorough grounding means that, where fundamental ideas have to be made clear, practical work and "making haste slowly" are essential to success. The necessary fundamental ideas, however, reduce themselves to two: the idea that fractional quantities are equivalent if both numerator and denominator have been multiplied or divided by the same number, and the idea that, for example, a third of one fifth is a fifteenth. On these two basic ideas rests the entire superstructure. Much practical work at this later stage, except to emphasise and make sound this twofold foundation, is a waste of time and unnecessary. Addition and subtraction of fractions can be speedily overcome if the first idea is clear; on it too depends the rule for division of fractions, while on the second idea multiplication of fractions is based. At the same time, fractions should, at first, never be used without concrete reference; children need not draw diagrams at this stage to show that $\frac{1}{3}$ yard, $+ \frac{2}{5}$ yard $= \frac{11}{15}$ yard, but they should be asked to find the length of $\frac{1}{3}$ yard $+ \frac{2}{5}$ yard, not at first merely the value of $\frac{1}{3} + \frac{2}{5}$.

We now make a few suggestions on the detailed teaching of fractions, in the hope that these may prove useful to teachers having difficulty in this part of their course.

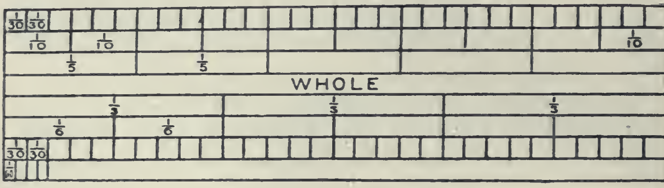
Equivalence of Fractions.

1. We may approach this from two illustrations. Since a whole yard consists of **5** fifths, or **10** tenths, or **15** fifteenths or **25** twenty-fifths, or **100** hundredths of a yard, so $\frac{1}{5}$ of a yard is equivalent to $\frac{2}{10}$ or $\frac{3}{15}$, or $\frac{5}{25}$, or $\frac{20}{100}$ of a yard. Or we may suppose a circular cake cut into **3** equal parts, and $\frac{2}{3}$ of it set aside to be given away. We ask ourselves, "To how many children can I give $\frac{1}{21}$ of this cake out of this $\frac{2}{3}$ cake?" If the whole cake be divided into **21** equal parts, obviously **7** of these go to each third, or $\frac{7}{21}$ cake $= \frac{1}{3}$ cake; therefore $\frac{14}{21}$ cake must equal $\frac{2}{3}$ cake. That is, in $\frac{2}{3} = \frac{14}{21}$, we have multiplied **3** by *seven* to get **21** and **2** by *seven* to get **14**, a mysterious rule unless explained by considering, not abstract fractions, but fractions of concrete quantities.

2. This idea will have received many illustrations in the work of the lowest standards, and first should come rapid revision of it as applied to halves, quarters, eighths, sixteenths; halves, thirds, sixths and twelfths; halves and tenths.

3. Each teacher will choose his own additional illustrations, but a selection of the following has proved useful in experience.

(i) Take paper ruled in squares as used in infant schools and lowest standards. Make an oblong **1** square deep and **30** squares across. This can be used to illustrate, say $\frac{2}{3} = \frac{4}{6} = \frac{20}{30}$ oblong or $\frac{3}{5} = \frac{6}{10} = \frac{18}{30}$ oblong, by making similar oblongs above and below the "whole," divided to show fifths, tenths and thirtieths, or thirds, sixths and thirtieths



These and similar papers are useful for reference in teaching addition or subtraction to dull children. They also illustrate, *e. g.* $\frac{1}{5}$ of $\frac{1}{6}$ oblong $= \frac{1}{30}$ oblong or $\frac{1}{3}$ of $\frac{1}{10}$ oblong $= \frac{1}{30}$ oblong. Children are keen to enlarge their diagrams to illustrate sixtieths, ninetieths and even smaller parts, thus themselves passing beyond the practical work to the theoretical idea. This individual work or questions such as "Find what part of the oblong $\frac{2}{3} + \frac{1}{5}$ oblong make, or $\frac{5}{6} - \frac{2}{5}$ oblong," will keep quick children busy while the slow ones complete the essential parts of their diagrams, as it is important that even slow children finish such work.

(ii) Take cardboard or real coins for 1 crown, 2 half-crowns, 5 shillings, 10 sixpences, 20 threepenny bits and make equivalent piles.

1 crown = 2 half-crowns = 5 shillings = 10 sixpences = 20 threepenny bits.

Express these equivalents as fractions of £1.

£$\frac{1}{4}$ = £$\frac{2}{8}$ = £$\frac{5}{20}$ = £$\frac{10}{40}$ = £$\frac{20}{80}$.

From this illustration children see clearly that the smaller the coin or part of £1, the higher the pile, that is, the greater the number of parts to be taken Here too they are quick to pass to the theoretical idea. An imaginary $1\frac{1}{2}d.$ coin is quickly diagnosed as £$\frac{1}{160}$ and £$\frac{40}{160}$ is given as the equivalent. They simply carry the teacher on to £$\frac{1}{320}$ or even £$\frac{1}{1280}$, revelling in fractions introducing such enormous numbers, but themselves so small.

(iii) Use letter-weighing balance to bring out this series of equivalents—

One 2-oz. = two 1-oz. = four $\frac{1}{2}$-oz. = eight $\frac{1}{4}$-oz., etc., passing, if possible, as before, beyond the actual weights:

i. e. $\frac{1}{8}$ lb. = $\frac{2}{16}$ lb. = $\frac{4}{32}$ lb. = $\frac{8}{64}$ lb. = $\frac{16}{128}$ lb.

4. When by illustrations of some kind the idea of the equivalence of fractional quantities has had a chance to develop, the rule can be made explicit by asking such questions as: "Fill in the blanks in $\frac{3}{7}$ yard = $\frac{}{14}$ yard; $\frac{6}{33}'' = \frac{2}{}''$; $\frac{20}{30}'' = \frac{}{3}''$; £$\frac{7}{9}$ = £$\frac{21}{}$." The results suggested can be tested diagrammatically and fully explained and the rule then formulated: "Form fractions equivalent to a given fraction by multiplying or dividing both upper and lower figures by the same number."

5. The value of the rule can be illustrated by such questions as "Find the value of $\frac{2}{3}$ oblong + $\frac{1}{5}$ oblong or of £$\frac{7}{20}$ + £$\frac{3}{80}$ + £$\frac{1}{160}$," to illustrate multiplication of both numerator and denominator, and "Which would you prefer, $\frac{22}{55}$ of £1 or $\frac{31}{62}$ of £1?" to illustrate division of both.

6. A very few such examples are sufficient as an introduction to an exciting drill lesson to cultivate speed and accuracy in changing denominators and reducing fractions to their lowest terms.

Addition and Subtraction.

Although reference to diagrams should be unnecessary for most pupils, a few backward children are certain to require it at some stage. With all pupils we should at first retain concrete reference, asking for the value, *e.g.* of $\frac{1}{3}$ in. + $\frac{5}{6}$ in., not of $\frac{1}{3} + \frac{5}{6} + \frac{2}{9}$. We suggest a simple subdivision of the work whereby many children find out each step for themselves. Addition and subtraction may be combined in most of these. Improper fractions and mixed numbers will arise, but a reference to fundamental ideas quickly overcomes any difficulty; *e. g.* there are seven sevenths in any whole, and therefore in $\frac{29}{7}$ there are 4 wholes and $\frac{1}{7}$. Reference to the concrete is now omitted, to save space.

(*a*) Same denominator: *e. g.* $\frac{5}{7} \pm \frac{2}{7}$.
(*b*) Common denominator suggested: *e. g.* $\frac{5}{6} \pm \frac{1}{2} \pm \frac{1}{12}$.
(*c*) A new denominator required, *e. g.* $\frac{1}{2} \pm \frac{1}{3}$.
Cases involving several harder numbers should be avoided till children can save time by using rule for finding L.C.M., *e. g.* $\frac{5}{9} + \frac{2}{7} + \frac{4}{15}$.
(*d*) Mixed numbers, *e. g.* $4\frac{1}{3} + 3\frac{1}{4} = 7\frac{4+3}{12} = 7\frac{7}{12}$.
N.B.—$7\frac{4+3}{12}$ is simply a short way of writing $7 + \frac{4}{12} + \frac{3}{12}$, one line and one common denominator serving for two or more fractions.

$$5\tfrac{1}{2} - 2\tfrac{1}{3} = 3\tfrac{3-2}{6} = 3\tfrac{1}{6}.$$

(*e*) Subtraction of mixed numbers, involving decomposition.

$$e.\,g.\ 11\tfrac{1}{3} - 4\tfrac{3}{7} = 7\tfrac{7-9}{21} = 6\tfrac{21+7-9}{21} = 6\tfrac{19}{21}.$$

Multiplication of Fractions.

1. For a discussion of the meaning of multiplication by a fraction see p. 54.

2. Multiplication of fractions by whole numbers is not difficult, and examples of it occur in the lower standards in connection with measurement.

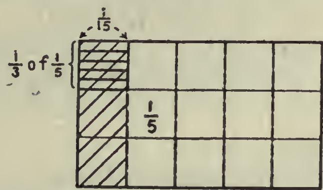
e. g. $2\frac{3}{4}d. \times 7 = 14d. + \frac{21}{4}d. = (14 + 5\frac{1}{4})d. = 1s.\ 7\frac{1}{4}d.$
$\frac{2}{3}$ yd. $\times 5 = \frac{10}{3}$yd. $= 3\frac{1}{3}$ yds., since in a yard there are 3 thirds of a yard.

3. Multiplication of a fraction by a fraction depends entirely on the idea underlying, say, $\frac{1}{3}$ of $\frac{1}{5} = \frac{1}{15}$. Note that "of" makes sense to a child where at first the multiplication sign confuses his ideas.

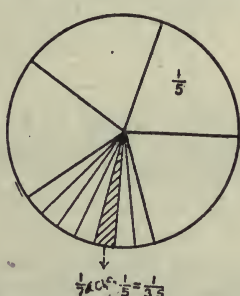
This should be illustrated until its meaning is fully grasped.

e. g. (*a*) Take a line of given length; show thirds of it; divide thirds into 4 equal parts. 12 of these small parts are in the whole. $\therefore \frac{1}{4}$ of $\frac{1}{3}$ line $= \frac{1}{12}$ line.

(*b*) Take a rectangle 5 units by 3 units on squared paper. Show fifths of this. Divide fifths into 3 equal parts. Thus $\frac{1}{3}$ of $\frac{1}{5}$ rectangle $= \frac{1}{15}$ rectangle.



(*c*) Take a circle; show fifths, and divide fifths into seven equal parts. Thus $\frac{1}{7}$ of $\frac{1}{5}$ circle $= \frac{1}{35}$ circle.



4. After this fundamental idea is familiar, we may advance in stages to the formulation and practice of the rule for multiplication of fractions; we suggest the following as having given good results—

(*a*) Set such a problem as, "$\frac{1}{3}$ of my garden is soil suited for growing vegetables. I sow $\frac{5}{7}$ of this with potatoes. What part of my garden do the potatoes occupy?"

Children should be able to make a good attempt at solution.

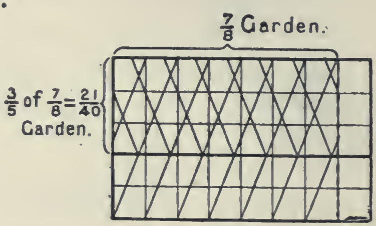
$$\tfrac{1}{7} \text{ of } \tfrac{1}{3} \text{ garden} = \tfrac{1}{21} \text{ garden (already known).}$$
$$\therefore \tfrac{5}{7} \text{ ,, } \tfrac{1}{3} \text{ ,, } = \tfrac{5}{21} \text{ ,,}$$

If we now alter in the problem $\frac{1}{3}$ and $\frac{5}{7}$ to $\frac{3}{5}$ and $\frac{7}{8}$, say, some of the class should solve this also.

$$\text{For } \tfrac{1}{5} \text{ of } \tfrac{1}{8} \text{ garden} = \tfrac{1}{40} \text{ garden}$$
$$\text{and } \tfrac{3}{5} \text{ ,, } \tfrac{1}{8} \text{ ,, } = \tfrac{3}{40} \text{ ,,}$$
$$\therefore \tfrac{3}{5} \text{ ,, } \tfrac{7}{8} \text{ ,, } = \tfrac{21}{40} \text{ ,,}$$

But slow children will find it difficult to grasp, and it is good for the whole class to make a diagram to show that this result is correct.

e. g.



Similar problems should be set until the children can work them by reasoning and illustrate their result by a diagram.

(*b*) If a list is kept of the answers so obtained, *e. g.* $\frac{3}{5}$ of $\frac{7}{8} = \frac{21}{40}$, and the question is asked, "Find, as quickly as you can, the value of $\frac{4}{7}$ of $\frac{5}{9}$ cake," many, looking at the results already obtained, will work this as $\frac{4 \times 5}{7 \times 9} = \frac{20}{63}$. When they are quite sure from discussion that this "multiplication" rule gives a sound result, they may be allowed to write $\times$ for "of," if they please, noting that $\times \frac{2}{3}$ has not exactly the same meaning as they have given to $\times$ **5**, say.

(*c*) The cancelling operation is best seen if an example involving large numbers is set: *e.g.* $\frac{7}{9}$ of $\frac{63}{77}$. Children will work this—

$$\frac{7}{9} \text{ of } \frac{63}{77} = \frac{7 \times 63}{9 \times 77} = \frac{441}{693} = \frac{147}{231} = \frac{21}{33} = \frac{7}{11}$$

applying their method of reducing a fraction to its lowest terms. They greatly appreciate the idea of taking out common factors before multiplying—

$$e.g.\ \frac{7}{9} \text{ of } \frac{63}{77} = \frac{7 \times \overset{\overset{1}{\cancel{7}}}{\cancel{63}}}{\underset{1}{\cancel{9}} \times \underset{11}{\cancel{77}}} = \frac{7}{11},$$

and soon will shorten this to—

$$\frac{\overset{1}{\cancel{7}}}{\underset{1}{\cancel{9}}} \text{ of } \frac{\overset{7}{\cancel{63}}}{\underset{11}{\cancel{77}}} = \frac{7}{11}.$$

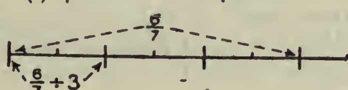
(*d*) Mixed numbers can then be introduced. Usually it is simplest to change these to improper fractions and proceed as in (*c*).

(*e*) Much drill is required to make multiplication sound and to gain speed and accuracy in cancelling, but this is congenial work to most children.

Division of Fractions.

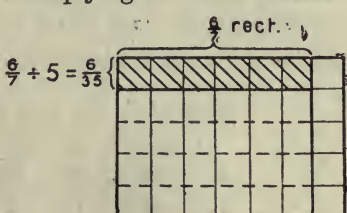
1. Easy examples may occur in earlier stages, involving division of a fraction by a whole number; these are of one of two types—

e.g. (i) $\frac{6}{7}$ line $\div 3 = \frac{2}{7}$ line.



(ii) $\frac{6}{7}$ rect. $\div 5 = \frac{6}{35}$ rect.,

multiplying the denominator.



2. The development of the ordinary rule for division by a fraction, "Invert the divisor and multiply," can be taken in stages. Before beginning, it is wise to tell the children that this is difficult work which leads in the end to a simple result.

(*a*) Make clear by easy examples that, if one quantity be divided by another, each must be expressed in the same unit; *e.g.* $\frac{6 \text{ pints}}{2 \text{ pints}} = 3$; $\frac{6 \text{ pints}}{2 \text{ pence}}$ makes no sense unless we know the value of a pint in pence. $\frac{6 \text{ shillings}}{4 \text{ pence}}$ is possible when shillings are changed to fourpences or pence.

(*b*) Give examples of this type: "How many $\frac{3}{4}$-lb. packets of tea can I make from $5\frac{1}{4}$ lb. tea?" Children will use a quarter of a pound as the new unit:

$$i.e.\ 5\tfrac{1}{4} \text{ lb.} \div \tfrac{3}{4} \text{ lb.} = 21(\tfrac{1}{4} \text{ lb.}) \div 3(\tfrac{1}{4} \text{ lb.}) = \tfrac{21}{3} = 7.$$

Lead on to examples where remainder occurs; *e.g.* $7\frac{2}{3}$ lb. $\div \frac{2}{3}$ lb. $= 23(\frac{1}{3}$ lb.$) \div 2(\frac{1}{3}$ lb.). This gives 11 whole packets each containing $\frac{2}{3}$ lb., and a single $\frac{1}{3}$ lb. left over, which is, however, half a packet.

$\therefore 23(\frac{1}{3} \text{ lb.}) \div 2(\frac{1}{3} \text{ lb.}) = 11\frac{1}{2}$ packets or 11 packets and $\frac{1}{3}$ lb. over.

Children tend at first to say $11\frac{1}{3}$ packets; writing division as $\frac{23}{2}$ helps to check this.

(*c*) Give examples where the new unit is not obvious but is suggested.

$$\begin{aligned} e.g. \quad & 6\tfrac{1}{2} \text{ pints} \div \tfrac{3}{4} \text{ pint} \\ &= 13(\tfrac{1}{2} \text{ pints}) \div 3(\tfrac{1}{4} \text{ pints}) \\ &= 26(\tfrac{1}{4} \text{ pints}) \div 3(\tfrac{1}{4} \text{ pints}) = \tfrac{26}{3} = 8\tfrac{2}{3} \end{aligned}$$

(*d*) Give examples where new unit has to be got from L.C.M.

$$\begin{aligned} e.g.\ 5\tfrac{1}{3} \text{ oz.} \div \tfrac{4}{5} \text{ oz.} &= \frac{16(\frac{1}{3} \text{ oz.})}{4(\frac{1}{5} \text{ oz.})} = \frac{16 \times 5(\frac{1}{15} \text{ oz.})}{4 \times 3(\frac{1}{15} \text{ oz.})} \\ &= \frac{80(\frac{1}{15} \text{ oz.})}{12(\frac{1}{15} \text{ oz.})} = \frac{80}{12} = \frac{20}{3} = 6\tfrac{2}{3}. \end{aligned}$$

(*e*) After a few examples of the previous type, suggest that children look for a quick way. Tabulate results already got from (*d*).

$$e.g.\ \tfrac{16}{3} \div \tfrac{4}{5} = \tfrac{16 \times 5}{4 \times 3}$$

and give a question like $\frac{7}{8}$ lb. $\div$ $\frac{2}{3}$ lb., asking for answer to be found as quickly as possible. Many will notice that, on analogy of others, it should be $\frac{21}{16}$ or $\frac{7 \times 3}{2 \times 8}$. The reason for this should be discussed, *i. e.* that each must be expressed in twenty-fourths of 1 lb.; since 8 goes into 24 3 times, we have 3×7 above, and since 3 goes into 24 8 times, we have 8×2 below. Take a few examples in the same way.

Let children then formulate the rule, comparing this division question with the multiplication one which would give the same answer.

(*f*) Give drill in the rule, introducing cancelling and mixed numbers as in multiplication.

CHAPTER X

DECIMAL FRACTIONS

THE development of decimal fractions is, historically, a recent one; the decimal fraction is a very young brother of the vulgar fraction, but he has proved himself so useful and in his essence so simple, that his position is assured. It would seem to be most natural to mankind to divide wholes into halves, quarters, eighths and sixteenths; into thirds, sixths and twelfths. The history of Arithmetic and the progress of a little child alike indicate this. But developing mankind and the developing child have found it very difficult to acquire the necessary mastery of harder vulgar fractions. Men, at one stage, tried to break up all such fractions into a sum of simple vulgar fractions and a serious operation they found this to be. Children found themselves obliged to add together, say, such monsters as $3\frac{47}{115}$ cwt., $4\frac{61}{186}$ cwt. and $19\frac{413}{620}$ cwt. To do this they had to find the L.C.M. of 115, 186 and 620; to find the L.C.M. of these numbers they had first to find their H.C.F.; to do this they had to learn a complicated rule for finding the H.C.F. of large numbers. Even when the way was clear, the individual steps were troublesome and errors easily crept in.

Both mankind and the child find these difficulties minimised by the use of the decimal fraction, that is, by expressing hard vulgar fractions as a sum of tenths, hundredths and thousandths, it being seldom necessary in practice to proceed to smaller subdivisions of the whole. It is easy by simple division to express any vulgar fraction as a sum of tenths, hundredths and thousandths; it is easy, having expressed vulgar fractions in such a way, to add, subtract, multiply and divide them to any required degree of accuracy. For these reasons, in school we now spend the time once devoted to gaining a mastery over fractional operations involving difficult denominators, in teaching the decimal notation for fractions and in extending the fundamental operations to this new form of fractional quantity. We do this, believing that the use of decimal fractions is easier than the use of vulgar fractions except in simple cases; believing too that because of their close connection with the metric system, our pupils must become familiar with decimal notation and calculation as a preparation for the practical affairs of life (see p. 115).

It follows that we begin our teaching of fractions by introducing our pupils to the simple vulgar fractions whose use in measurement comes naturally to children (see p. 121). At a later stage, in Standards IV. and V., we have to develop the fundamental operations for vulgar fractions, but, if wise, we shall restrict ourselves, especially in addition and subtraction, to fractions with easy denominators and in frequent use, thus avoiding the necessity of teaching the long H.C.F. rule and avoiding, too, many wearisome hours over struggles with fractions. Instead of this, we shall teach the application of the fundamental operations to decimal fractions.

The decimal notation, however, like the notation for vulgar fractions, requires time to fix itself in our pupils' minds, so that the earlier we can introduce it the better for the later development of decimal operations. Although plans will vary from school to school, we suggest the following as a suitable method. Children in Standards I. or II. can use the first decimal place in connection with measurement in tenths of an inch, and there is no reason to avoid simple addition, subtraction, multiplication and division questions if they arise naturally in connection with such measurement and are based on practical work. In Standard III. they can make the acquaintance of, at least, centimetres in the metric system; there seems good reason often for its further development. This will again illustrate the use of tenths and the simple operations can be practised as far as the first decimal place; *i. e.* addition, subtraction, multiplication and division by units. In Standards III. or IV. the notation can be extended to two decimal places by the use of squared paper, which gives an admirable illustration of 1 sq. in., ·1 sq. in., ·01 sq. in., and of tenths of tenths being hundredths; while leading naturally to the decimalisation of money as far as shillings. The metric system when extended in these standards gives a further illustration of the second decimal place and introduces the third place. Easy decimal operations can all be practised in connection with this new knowledge of decimal notation. In Standard IV. or V. a full revision of our Arabic notation, bringing out the contrast between it and, say, Roman notation (see p. 40), and emphasising the simplicity of its basis in place value, is a wise introduction to a complete grasp of the decimal notation to any number of places. Addition and subtraction are easily extended; multiplication and division by any kind of whole number or decimal fraction require more careful treatment. Decimalisation of money should be extended to three places, so that a beginning can be made to the use of decimals in general work. Still farther on in school, it is often right to develop methods of multiplication and division by decimals, contracted so as to avoid unnecessary figures and yet give results to a required degree of accuracy. The necessary considerations for addition and subtraction are so simple that they might well be taken in most schools, because, along with the extension of decimalisation of money to five places, they afford a simple method for calculation of prices, based on the idea of "practice." We shall illustrate this later.

We propose to treat the subsequent work in decimals, then, under the following heads—

1. Introductory Work in Lower Standards.
2. The Extension of the Fundamental Operations to Decimals.
3. Decimalisation of Money.
4. Contracted Methods for Decimals, with Examples of their Application to Practice and Compound Interest.

Introductory Work in Lower Standards.

We shall merely suggest a few pieces of work which have, in practice, familiarised children with decimal notation and introduced easy applications of the fundamental operations to decimal fractions.

(*a*) Measurement of lines in inches and tenths of an inch.

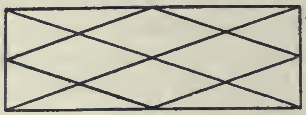
Drawing of lines of given length: finding sums and differences of these lines (*e.g.* $4{\cdot}3'' \pm 2{\cdot}9''$); finding, by drawing, say, 5 times $1{\cdot}3''$; dividing a line of given length into a certain number of equal parts, *e.g.* $6{\cdot}9'' \div 3$, $4{\cdot}5'' \div 3$.

Making of objects in stiff paper or thin card, using measurements in inches and tenths of an inch: *e.g.* block letters, match box, pin tray.

Questions on objects made to introduce easy operations: *e.g.* How far would a fly walk in going from one point to another on this object? How much longer is this side than that? If 6 of you placed your objects side by side, how long a row would they make? If you made another, every part to be $\frac{1}{3}$ of the first one, how long would each edge be?

(*b*) Measurement in centimetres and millimetres.

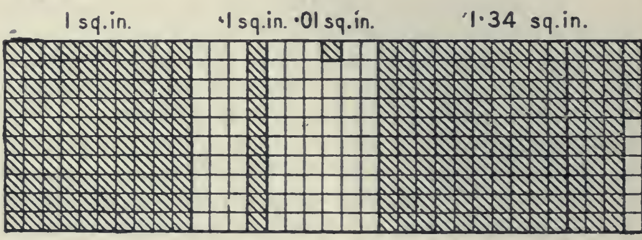
Well-known facts about diagonals of rectangles, sides and diagonals of parallelograms, equilateral and isosceles triangles, regular hexagons, discovered by measurement: *e.g.* Draw an oblong, sides 5·6 cm. and 7·4 cm. Halve each of its sides and join the points found. Measure the new lines and see if you can find out anything about them. Try to explain why they are all the same length. How much farther is it to go round your first four-sided figure than your new one? Draw both the diagonals of your oblong. What do you notice? Point out any other journey on your oblong which would be the same length as a walk along 1 diagonal and back again.



Such work can obviously be extended and it has the merits of giving variety, of demanding measurements of which the result is known by the teacher, and of developing geometrical notions.

(*c*) Drawing of diagrams on squared paper, where 1 sq. in. is the unit, a column of 10 little squares ·1 or $\frac{1}{10}$ unit, and 1 little square ·01 or $\frac{1}{100}$ unit.

e. g.



Writing down size of area shaded in: *e. g.* 1·52, 2·04 sq. in.

Shading in given areas, *e. g.* 1·41, 2·03, ·07, ·3 sq. in. Finding total area shaded in, difference in area between two shadings, an area three times or a quarter of a given area, thus developing the fundamental operations to two decimal places.

If diagram represents 1 sq. in. of gold valued at £1, finding what ·1 sq. in. represents; shading of gold worth £1·3, £1 8*s*., £2 4*s*., 6*s*.: finding total area shaded in and its value; shading of gold worth 1*s*., 7*s*.: finding value of areas 2·35, 4·75, 2·05 sq. in., thus developing decimalisation of money as far as shillings.

Shading ·1 sq. in., more heavily ·1 of ·1 sq. in., and expressing result as a decimal of a square inch; shading ·5 sq. in., ·1 of ·5 sq. in., and expressing result as a decimal; shading ·1 sq. in., ·4 of ·1 sq. in.; then ·2 of ·3 sq. in., ·4 of ·5 sq. in., ·3 of ·7 sq. in.; expressing all results as decimals of a square inch and thus developing the idea that tenths of tenths give hundredths.

Shading $\frac{1}{10}$ of a tiny square, and asking such questions as: What part is this of a square inch? How might it be written? Show as well as you can ·005 sq. in. Of what is this the half? Show ·05 sq. in. Of what is this the half? What is half of 1 sq. in.; of ·1 sq. in.; of ·01 sq. in.; of ·001 sq. in.? Show the answer to the last on your diagram.

This piece of work carries on the child's mind to the further development of decimal notation and we may be content if it simply gives him a prospect without leading to any very definite knowledge. Some use a square of 10″ side as a unit, and by so doing show decimals up to fourth decimal place very easily.

(*d*) Measurement by the metric system.

See Chapter VIII. for detailed treatment. Here the child can practise knowledge of decimal notation already gained, as the work provides plenty of easy decimal manipulation, besides affording a further illustration of the relation of tenths to hundredths and thousandths. The third decimal place thus becomes familiar to the child.

If some such work as the above is developed in Standards I. to IV., there is no reason why the full treatment of decimals should present any undue difficulty to a bright Standard IV. or an average Standard V.

Extension of Fundamental Operations to Decimal Fractions.

1. Before a full study of decimals is undertaken, it is well to revise our general notation, bringing out its basis in

place value, and thus showing the decimal notation as its natural extension. Some hints will be found in Chapter I. on Notation; children at this stage profit by some acquaintance with the historical development of notation and can be led to appreciate the simplicity and usefulness of a notation based on place value.

2. Addition and subtraction need no comment. If the first questions arising in the lower standards have been worked in connection with measurement, children have already grasped that tenths are added by themselves, hundredths by themselves, just as units, tens, and hundreds are treated separately. They also know that ten tenths make a whole, every ten hundredths a tenth, every ten thousandths a hundredth. These ideas can be extended to any number of decimal places without difficulty.

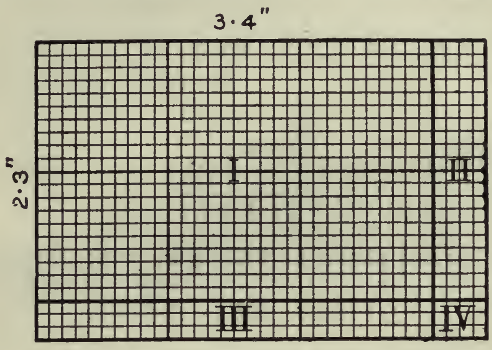
3. The first steps in multiplication are equally easy. Multiplication by a unit makes no alteration in the value of the decimal; *e.g.* 7 times 3 hundredths is simply 21 hundredths, usually written as 2 tenths and 1 hundredth or ·21; there is no new principle either here or in multiplication by a whole number which is easily factorised.

But when we come to multiplication of a decimal quantity by a decimal quantity we find difficulties. Three distinct methods are possible, the question of the ideal method probably depending on the conditions and complete scheme of the school concerned. In any case the same introductory treatment is wise, consisting of some such work as was outlined in (*c*) of previous section, followed by diagrammatic work of greater difficulty.

e.g. A rectangle measures 2·3″ by 3·4″. Find its area. This should be worked by drawing and also by calculation.

$$\begin{array}{rl} \text{Area} = & \text{I} + \text{II} + \text{III} + \text{IV}. \\ \text{I} = & 3 \times 2 = 6 \quad \text{(sq. in.)} \\ \text{II} = & \cdot 4 \times 2 = \cdot 8 \quad \text{(sq. in.)} \\ \text{III} = & 3 \times \cdot 3 = \cdot 9 \quad \text{(sq. in.)} \\ \text{IV} = & \cdot 4 \times \cdot 3 = \cdot 12 \text{ (sq. in.)} \\ \hline & \therefore 2\cdot 3 \times 3\cdot 4 = 7\cdot 82 \end{array} \qquad \begin{array}{ll} 3\cdot 4 & \\ 2\cdot 3 & \\ \hline 6\cdot 8 & (\text{I} + \text{II}) \\ 1\cdot 02 & (\text{III} + \text{IV}) \\ \hline 7\cdot 82 & \end{array}$$

A few examples of this kind will lead to a natural development in the child's mind of the fact that tenths



multiplied by tenths give hundredths, leading on to tenths of hundredths and hundredths of tenths giving thousandths, and so to the simple rule of counting the sum of the numbers of decimal places in the factors to find the number of decimal places in the product.

Let us look now at the possible methods—

(*a*) A few people advocate multiplication of the quantities as given, applying first principles continuously.

e.g. £4·23 × ·605.

$$\begin{array}{rl} \pounds 4\cdot 23 & \\ \cdot 605 & \\ \hline 2\cdot 538 & = \pounds 4\cdot 23 \times \cdot 6 \\ \cdot 02115 & = \pounds 4\cdot 28 \times \cdot 005 \\ \hline \pounds 2\cdot 55915 & = \pounds 4\cdot 23 \times \cdot 605. \end{array}$$

This method never leads to a mechanical rule; each question must be thought out afresh. The result is that it is unsuitable for the ordinary type of child in our Elementary Schools, though it is occasionally good for children in upper standards to check their work by such a method. It may be said, too, that even the cleverest folk find a mechanical method better; it saves their brains for more important matters!

(*b*) Many multiply decimal quantities as if they were whole numbers and then, after multiplication, insert the point in its correct position—

e.g. £4·23 × ·605

$$\begin{array}{r} 423 \\ 605 \\ \hline 2538 \\ 2115 \\ \hline 255915 \end{array}$$

Two decimal places in **4·23**, **3** in **·605**; therefore **5** in product.

∴ **£4·23 × ·605 = £2·55915.**

Method is on similar lines if one prefers to multiply first by units rather than hundreds.

The explanation is of the simplest. In changing **4·23** to **423** we have made our product **10** times **10** times too big; in changing **·605** to **605** we have made it **10 × 10 × 10** times too big. Thus, altogether our result is **10 × 10 × 10 × 10 × 10** or 10^5 times too large. It is, therefore, necessary to divide the resulting product by **10**, five times, a process which gives us five decimal places.

It is quite possible to teach this sensibly to children. They are asked, say, to find the cost of **2·3** yards of cloth at **4·4***s*. a yard. They could do it if it were **23** yards at **44***s*. Let them work this and criticise their answer. They will see that they have taken **10** times too much cloth and **10** times too high a price, and therefore the total cost is **100** times too great. The two decimal places which result from dividing total cost by **100**, agree with their previous knowledge that tenths by tenths give hundredths and that an area (**5·7 × 4·3**) sq. in. gives an answer involving two decimal places.

They can proceed to extend this, say, to **2·3 × ·44**, noting that the result is **1000** times too great. After taking a few examples of various kinds and reasoning about them thoroughly, we may ask for a rule; when given, we can elicit an explanation of it from our class. The rule may then be applied in general. It is so mechanical and so simple that it gives admirable results in speed and accuracy. It has, however, one serious defect; it in no way prepares the child to shorten his multiplication by omitting unnecessary figures. This is because during the actual multiplication the value of the quantities is neglected; hence no one can see what figures may safely be omitted, what must be retained. In general, however, this is the simplest method, except in schools where a great proportion of the pupils will in time be learning contracted methods of dealing with multiplication of decimals.

(*c*) A third method has been developed which is a compromise between (*a*) and (*b*). It preserves the value of each figure in the product as (*a*) does and as (*b*) does not; yet it lends itself to mechanical work as (*b*) does and as (*a*) does not. The method is that of expressing the multiplier in what is known as "the standard form," that is, as a number whose first figure is in the units place.

e. g. £·42 × ·64 = £·042 × 6·4.

$$\begin{array}{l} \underline{\pounds\cdot042 \quad \times 6\cdot4} \\ \pounds\cdot252 \quad = \pounds\cdot042 \times 6 \\ \underline{\pounds\cdot0168 = \pounds\cdot042 \times \cdot4} \\ \pounds\cdot2688 = \pounds\cdot042 \times 6\cdot4 = \pounds\cdot42 \times \cdot64. \end{array}$$

Express one factor in standard form to serve as multiplier; alter the decimal point in the other factor exactly the same number of places in the opposite direction (*i. e.* multiply by some power of 10 to cancel effect of division by same power of 10 or divide by some power of 10 to cancel effect of multiplication by the same power of 10). Then multiply by units figure first. The result lies immediately underneath multiplicand, since multiplication by a unit does not alter place value. Other figures in multiplier follow the usual multiplication rule—one place farther to the right each time. The resulting product needs no alteration.

Children can reach this easily if led along by easy steps.

(i) £4·3, 5·21″, 72·319 metres × 2 or 3 or 7 to emphasise fact that multiplication by units figure gives decimal point in product underneath decimal point in multiplicand—

$$e.g.\quad \frac{72{\cdot}319 \text{ metres} \times 3}{216{\cdot}957 \text{ metres}}$$

(ii) £4·3, 5·21″, 72·319 metres × 2·1 or 3·2 or 7·43 to bring out, by reasoning, the fact that subsequent figures in multiplier move result each time one place towards left—

$$
\begin{array}{lll}
e.g. & \underline{5\cdot21'' \quad \times 7{\cdot}43} & \\
 & 36\cdot47 & = 5{\cdot}21'' \times 7 \\
 & 2\cdot084 & = 5{\cdot}21'' \times {\cdot}4, \text{ since } \tfrac{4}{10} \text{ of } \tfrac{1}{100} = \tfrac{4}{1000}. \\
 & \cdot1563 & = 5{\cdot}21'' \times {\cdot}03, \text{ since } \tfrac{3}{100} \text{ of } \tfrac{1}{100} = \tfrac{3}{10000}.
\end{array}
$$

Rule is obviously the ordinary rule for multiplication.

(iii) £4·3, say, × ·21.

Children know what they would like multiplier to be; *i.e.* 2·1. They can see that if they work with 2·1 the answer will be 10 times too large, and find out one way of correcting this by dividing £4·3 by 10, giving £·43. They then multiply £·43 by 2·1 as above.

$$
\begin{array}{r}
\underline{\pounds\cdot43 \times 2\cdot1} \\
\cdot86 \\
\underline{\cdot043} \\
\pounds\cdot903
\end{array}
$$

They can prove that the result is right by working £4·3 × ·21 by method (*a*), noting the advantage of a mechanical rule.

(iv) After a few such examples the rule for reduction to standard form may be formulated and definitely practised.

This method can be taught sensibly to children and each figure in the product is in its correct position; the method quickly becomes mechanical, if not so quickly as (*b*); it is a suitable method for later contracted work.

For these reasons it is worth teaching in those elementary schools in which a great proportion of the children will later learn contracted multiplication. In other schools probably the greater simplicity of (*b*) justifies its retention.

4. Division by units and by factors is, like multiplication, very easy, if notation is understood. The rule is simple. Division by a unit does not alter place value of quantity; *e. g.* 21 hundredths divided by 3, simply gives 7 hundredths.

But when we come to division of a decimal quantity by a decimal quantity, again we find difficulties and again three distinct methods are possible, each parallel to the corresponding method in multiplication.

(*a*) A few people divide the quantities as given, continuously applying first principles.

e. g. 54·4641 kilometres ÷ ·126 kilometre.

```
      432·2
    ─────────
·126)54·4641
      50·4
      ────
       4·06
       3·78
       ────
        ·284
        ·252
        ────
        ·0321
        ·0252
        ─────
         0069
```

5 tens ÷ 1 tenth gives hundreds. ∴ first figure is placed in hundreds place. After that, working follows ordinary division rule and can be explained fully step by step, *e.g.* 4·06 ÷ ·126. 4 units ÷ 1 tenth gives tens; ∴ second figure is placed in tens place.

The result might be given as: 432·2 lengths of ·126 km and ·0069 km. left over. The remainder is obviously correct as above, since each figure in the working retains its correct value. But the method never leads to a mechanical rule and so is unsuitable for general use in school. See comment on corresponding method for multiplication.

(*b*) Many divide one decimal quantity by another only after changing the divisor into a whole number. A few also change the dividend into a whole number, but this causes unnecessary confusion.

e. g. 54·4641 km. ÷ ·126 km.

= 54464·1 ÷ 126, measuring each in thousandths of a kilometre.

```
          . . 4 3 2 · 2
        ---------------
1 2 6 ) 5 4 4 6 4 · 1
        5 0 4
        -----
          4 0 6
          3 7 8
          -----
            2 8 4
            2 5 2
            -------
              3 2 · 1
              2 5 · 2
              -------
                6 · 9
```

As in short division first figure goes above furthest figure required for first division; for

126 goes into 5 10,000's, 0 10,000's
126 „ 54 thousands, 0 thousands
126 „ 544 hundreds, 4 hundreds.

To justify position of point before last figure, we have—

126 goes into 321 tenths, 2 tenths :
∴ decimal point before 2 in quotient.

Remainder = 6·9 ÷ 1000, since question has been worked with the value of each figure 1000 times its actual value, *e. g.* worked in metres instead of kilometres.

Result is 432·2 lengths each ·126 km. long, and 6·9 metres or ·0069 km., left over.

This is much the simplest rule for division by a decimal quantity, since the position of the first figure in the quotient follows the same rule as for short division. The value of the remainder is seldom required, so that the necessity for occasionally altering it need not be counted a serious drawback. A much more serious one is that since each figure loses its true place value, a contracted method of division cannot easily be based on it; we cannot see which figures can safely be omitted.

This can be taught sensibly to children. After dividing by units and by numbers which can be factorised into units, they proceed to divide by, say, 26, or 131, and find that except for writing out products and remainders fully, the procedure is exactly similar and no trouble as to position of decimal place arises. They can after a time be asked to divide by ·26. They will be puzzled and can be asked instead to divide by 26 and criticise their result. It will be, of course, 100 times too little. They can put this right either by altering quotient after working or by altering dividend before working, in each case multiplying by **100**.

After taking a few similar examples and reasoning fully concerning them, the rule can be developed that we move the decimal point same number of places in same direction in dividend and divisor. Compare with equivalence rule for vulgar fractions, when division is written in fractional form—

$$\frac{54{\cdot}4641}{{\cdot}126} = \frac{54{\cdot}4641}{{\cdot}126} \times \frac{1000}{1000} = \frac{54464{\cdot}1}{126}$$

The remainder can be discussed in connection with the above questions as fully as may be desired in individual circumstances, but it is comparatively unimportant in division of decimals.

(*c*) A third method for division of decimals has been developed which is a compromise between (*a*) and (*b*). It leads to a mechanical rule, as (*b*) does and as (*a*) does not; it leads to a simple rule for contracted division, as (*a*) does and as (*b*) does not. It consists in expressing the divisor in "standard form," so that our divisor is always approximately some units figure; the process is thus closely connected with short division by a units figure.

e.g. **54·4641** km. ÷ **·126** km.
= **544·641** ÷ **1·26**, measuring each in tenths of a kilometre.

```
        4 3 2 · 2
1 · 2 6)5 4 4 · 6 4 1
        5 0 4
          4 0 · 6
          3 7 · 8
            2 · 8 4
            2 · 5 2
              · 3 2 1
              · 2 5 2
              · 0 6 9
```

Divisor is approximately **1**.

Position of first figure in quotient—

5 hundreds ÷ **1** (a unit) gives hundreds; ∴ place first figure above first figure in dividend.

Had divisor been greater than **5**, we should have had **54** tens ÷ some unit greater than **5**, which gives tens; ∴ first figure would have gone above second figure in dividend. Cp. its position in short division.

Remainder = ·**069** ÷ **10** because every figure in the working has ten times its actual value.

Children can be taught this sensibly, if we base the work on short division—

(i) **£6·93, £16·25, £24·31 ÷ 3, 5, or 7,**
to emphasise position of first figure under either first or second figure in dividend and to introduce idea of proceeding to any number of decimal places.

(ii) **£1·95 ÷ 1·5, £5·52 ÷ 2·4, £3·285 ÷ 7·3,**
to bring out fact that position of first figure in quotient is determined by the divisor being approximately **1** or **2** or **7**, but its value will be affected to some extent by the carrying figure from the tenths. This stage of finding and placing correctly the first figure in the quotient is not easy and requires much time before a further advance can be made.

(iii) **£·01426 ÷ ·031** or **£·943 ÷ ·41.**
Children know that they could deal with **3·1** or **4·1** as divisor. Let them work sum thus and criticise result, discovering that they must treat either dividend or quotient in same way as they treated divisor. See note on teaching of method (*b*).

(iv) After a few examples the rule for reduction to standard form can be formulated and definitely practised.

This rule, it must be confessed, does not give as good results in accuracy as the corresponding method for multiplication of decimals, the real trouble arising in stage (ii) as noted above. With average children its results are much worse than those obtained from method (*b*). But in schools where almost every child has later to deal with contracted methods for division, it is most probably worth while to give to division at this stage the extra time necessary to teach method (*c*) instead of method (*b*).

N.B.—In multiplication and division of decimals long and complicated sums should never be set to children, as adults work them either by tables or by use of contracted methods.

5. In dealing with the relationship of vulgar and decimal fractions no serious difficulties arrive. Obviously, if children understand decimal notation,

$$\cdot 35 = \frac{35}{100} = \frac{7}{20} \text{ or } 2\cdot 137 = 2\frac{137}{1000}$$

is straightforward work. The common equivalents should, of course, be known by heart; *i.e.* $\frac{1}{2}$'s, $\frac{1}{4}$'s, $\frac{1}{8}$'s, $\frac{1}{3}$'s, $\frac{1}{5}$'s and $\frac{1}{10}$'s.

In changing from vulgar fractions to decimals, the process is simply one of division and in most cases only approximate results are necessary—

e. g. $\frac{63}{85} = \cdot 7$ approx. or $\cdot 74$ approx. by dividing 63 by 85.

The difficulty of recurring decimals is now avoided by the use of approximate and contracted methods. Children need only learn the meaning of their notation when it arises naturally—

e. g. $\frac{1}{3} = \cdot 3333$............which is written $\cdot\dot{3}$
or $\frac{1}{6} = \cdot 166666$.......... „ „ „ $\cdot 1\dot{6}$
or $\frac{5}{11} = \cdot 45454545$........ „ „ „ $\cdot\dot{4}\dot{5}$

Decimalisation of Money.

Various methods are in common use. The essential is to choose some form which can be taught intelligently and worked mentally with speed and accuracy. We commend the following as one which has proved successful in practice and which satisfies the above essentials. But it is by no means the only possible method. We shall briefly indicate a simple method of developing it, accompanied by graded examples fully set down. The sooner our pupils set down nothing save the result, the better we may count the results of our teaching.

A. Direct Decimalisation of Money.

(1) Consider, say, £3 7*s.* 10*d.*, which is expressed as $\pounds 3 + \pounds\frac{7}{20} + \pounds\frac{10}{240}$. We require a method of expressing it as $\pounds 3 + \pounds\frac{\ }{10} + \pounds\frac{\ }{100} + \pounds\frac{\ }{1000}$: therefore we look for equivalents.

(2) £$\frac{1}{10}$ = 2s.
  ∴ £6 4s. = £6·2; £8·4 = £8 8s.
  Anything less than 2s. will not affect the first decimal place.

(3) £$\frac{1}{100}$ would be about 2½d., which is too awkward an equivalent to be considered.

(4) £$\frac{1}{1000}$ is very nearly ¼d. Try to find exact relationship between these. To save tongue-labour ("thousandths" being a difficult word), £$\frac{1}{1000}$ or £·001 may be called a "mil."

Thus, by definition £1 = 1000 mils.
But £1 = 960 farthings.
∴ 960 farthings = 1000 mils.
i.e. 24 farthings = 25 mils.
or 1 farthing = $(1 + \frac{1}{24})$ mil.
i. e. each farthing = £$\frac{1}{1000}$ + $\frac{1}{24}$ of £$\frac{1}{1000}$

∴ £6 4s. 2¼d. = £6·2 + 9 farthings.
= £6·2 + $9\frac{9}{24}$ mils.
= £6·209 + ⅜ of a thousandth of £1.
= £6·209 . . . to three places.
or = £6·20937 . . . to five places (⅜ = ·375).

£3 16s. 4d. = £3·8 + 16 farthings.
= £3·8 + $16\frac{16}{24}$ mils.
= £3·817 . . . to three places.
or = £3·81667 . . . to five places (⅔ = $\cdot\dot{6}$).

£4 8s. 1¾d. = £3·407 + $\frac{7}{24}$ mil.
= £3·407 + $\frac{1\cdot75}{6}$ mil. (7 = 4 × 1·75).
= £3·407 + $\cdot2\dot{9}$ mil.
= £3·407 . . . to three places.
or = £3·40729 to five places.

(5) But from florins to farthings is too great a gap. The above examples cover only cases of even number of shillings and odd pence less than 6d. To make system complete, we must take account of 1s. 6d. and 1s. and 6d. Since children make many mistakes over decimal places when these are introduced in the calculation as decimals of £1, the ex-

periment of treating them as groups of mils has been tried and found to be successful—

$$\begin{aligned}
i.e.\text{ since } 2s. &= £\cdot 100 \text{ or } 100 \text{ mils.}\\
1s.\ 6d. &= \tfrac{3}{4} \text{ of } 100 \text{ mils.}\\
&= 75 \text{ mils.}\\
1s. &= 50 \text{ mils.}\\
6d. &= 25 \text{ mils.}
\end{aligned}$$

$$\begin{aligned}
\therefore £8\ 17s.\ 5\tfrac{1}{4}d. &= £8\cdot 8 + 1s. + 21 \text{ f.}\\
&= £8\cdot 8 + (50 + 21\tfrac{21}{24}) \text{ mils.}\\
&= £8\cdot 871 + \tfrac{7}{8} \text{ mils.}\\
&= £8\cdot 872 \text{ to three places.}\\
\text{or } &= £8\cdot 87187 \text{ to five places } (\tfrac{7}{8} = \cdot 875).
\end{aligned}$$

$$\begin{aligned}
£3\ 7s.\ 10\tfrac{1}{2}d. &= £3\cdot 3 + 1s.\ 6d. + 18 \text{ f.}\\
&= £3\cdot 3 + (75 + 18\tfrac{18}{24}) \text{ mils.}\\
&= £3\cdot 394 \text{ to three places.}\\
\text{or } &= £3\cdot 39375 \text{ to five places.}
\end{aligned}$$

$$\begin{aligned}
£3\ 6s.\ 10\tfrac{1}{4}d. &= £3\cdot 3 + 6d. + 17 \text{ f.}\\
&= £3\cdot 3 + (25 + 17\tfrac{17}{24}) \text{ mils.}\\
&= £3\cdot 343 \text{ to three places.}\\
&= £3\cdot 34371 \text{ to five places } (\tfrac{17}{24} = \tfrac{4\cdot 25}{6} = \cdot 71).
\end{aligned}$$

(6) *Summary.*

(i) Take even number of shillings as florins—

1 florin = £·1.

(ii) Take largest group possible of—

$$\begin{aligned}
1s.\ 6d. &= 75 \text{ mils.}\\
1s. &= 50 \text{ mils.}\\
6d. &= 25 \text{ mils.}
\end{aligned}$$

(iii) Take remaining pence and farthings as farthings, each one being equal to $1\frac{1}{24}$ mil.

(iv) (i) gives first decimal place : (ii) and (iii) added together give second and third decimal places Twenty-fourths of a mil give the fourth and fifth decimal places, when these are wanted.

B. Inverse Process.

(1) *Summary.*

(i) Approximate to three decimal places.
(ii) Take tenths as florins.
(iii) Take away the largest possible group of—

75 mils = 1*s.* 6*d.*
50 mils = 1*s.*
25 mils = 6*d.*

(iv) Take the remaining mils as farthings, noting that—

$$1 \text{ mil} = (1 - \tfrac{1}{25}) \text{ farthings.}$$

For, 1000 mils = 960 farthings = £1

$\therefore$ 25 mils = 24 f.

$\therefore$ 1 mil $= \frac{24}{25}$ f.

$= (1 - \frac{1}{25})$ f.

Where number of mils lies between 1 and 12, the error in calling a mil a farthing varies from $\frac{1}{25}$ to $\frac{12}{25}$, which is less than half a farthing and therefore negligible.

Where the number of mils lies between 13 and 24, the error varies from $\frac{13}{25}$ to $\frac{24}{56}$, which is greater than half a farthing and must therefore be corrected by subtracting 1.

(2) *Graded examples.*

(i) £3·4 = £3 8*s.*

(ii) £3·411 = £3 8*s.* + 11 mils.
= £3 8*s* + $(11 - \frac{11}{25})$ f.
= £3 8*s.* $2\frac{3}{4}d.$ to nearest farthing.
or = £3 8*s.* 3*d.* to nearest penny.

(iii) £3·413 = £3 8*s.* + $(13 - \frac{13}{25})$ f.
= £3 8*s.* + 12 f. approximately.
= £3 8*s.* 3*d.*

(iv) £3·358 = £3 6*s*. + 50 mils + 8 mils.
= £3 7*s*. + 8 f. approximately.
= £3 7*s*. 2*d*.

(v) £3·791 = £3 14*s*. + 75 mils + 16 mils.
= £3 15*s*. 6*d*. + $3\frac{3}{4}$*d*. approximately.
= £3 15*s*. $9\frac{3}{4}$*d*. to nearest farthing.
or = £3 15*s*. 10*d*. to nearest penny.

(vi) £4·544 = £4 10*s*. + 25 mils + 19 mils.
= £4 10*s*. 6*d*. + $4\frac{1}{2}$*d*. approximately.
= £4 10*s*. $10\frac{1}{2}$*d*. to nearest farthing.
or = £4 10*s*. 11*d*. to nearest penny.
because $(19 - \frac{19}{25})$ f. is nearer 20 f. than 16 f.

(vii) £4·86173 = £4·862 to nearest mil.
= £4 17*s*. 3*d*. approximately.

Contracted Methods.

1. Addition.

To find an approximate answer, correct to any required decimal place, two additional places are required for addition. For—

(i) the carrying figure in the first additional place may alter considerably the required place:

(ii) the carrying figure in the second additional place may alter the figure in the first decimal place from below 5 to above 5 and so alter the required place by 1.

Example.

Find the sum of 5·3993 Km., 2·7248 Km., 8·6837 Km., 5·4584 Km., to the nearest first decimal place.

(*a*) Long method.

$$\begin{array}{r} 5\cdot3993 \\ 2\cdot7248 \\ 8\cdot6837 \\ 5\cdot4584 \\ \hline 22\cdot2662 \end{array}$$

∴ Answer required is 22·3 Km.

(*b*) Omission of all additional places.

5·3
2·7
8·6
5·4
22·0

This is wrong, the error in the required place being 3, mainly due to omission of carrying figure from second decimal place.

(*c*) Inclusion of one additional place.

5·39
2·72
8·68
5·45
22·24

This gives 22·2, the error in the required place being 1, due to omission of carrying figure from third decimal place causing inaccuracy in second decimal place. In general a more accurate result can be obtained by approximating the first additional figure, but this is rather too frequently a risk.

(*d*) Inclusion of two additional places.

5·399
2·725
8·684
5·458
22·266

This gives 22·3, which is correct. It is possible to imagine a sum of many rows where the inclusion of two additional places, even with the safeguard of approximating the last place, as in lines 1 and 3 above, still leaves an error of 1. But in contracted methods we aim at a good practical rule. Some people think plan (*c*) above, made more accurate by approximating the last place, is sufficiently safe for general purposes. We prefer ourselves the additional safety ensured by (*d*) and shall extend the exposition to multiplication on this plan.

2. Multiplication.

Since we have to add the various partial products to find the complete product, it will be necessary to work always with two additional places so that the result of the addition may be sufficiently accurate. All figures of less place value may be omitted throughout.

e.g. Find the value of 43·713928 × ·483615 to two decimal places.

Changing to "standard form" we have—

$$4 \cdot 3713 \mid \cancel{9}\cancel{2}\cancel{8} \times 4 \cdot 83615$$

$$\begin{array}{r|}
17 \cdot 4856 \\
3 \cdot 4970 \\
\cdot 1311 \\
262 \\
4 \\
2 \\
\hline
21 \cdot 1405
\end{array}$$

Result : 21·14

(i) Work to 2 + 2, *i.e.* four decimal places. Draw line after $\frac{3}{10000}$; figures beyond are unnecessary.

(ii) 928 at end is therefore not required. Consider carrying figure from 4 × 9, however. 4 × 9 = 36 : carry 4. 4 × 3 = 12; 12 + 4 = 16.

(iii) First line goes immediately underneath multiplicand. Each succeeding product comes one place further to the right, *i.e.* one place further beyond line barrier; therefore each time one figure can be cut off from multiplicand and considered only to find carrying figure for fourth decimal place, *e.g.* in line (2),

8 × 3 = 24 : carry 2
8 × 1 = 8; 8 + 2 = 10.

3. Division.

Since addition does not enter into this operation, two additional places need not be considered. We must, however, know one additional place beyond the required figure : *e.g.* ·04*4* = ·04 approximately and ·04*6* = ·05 approximately.

Consider the following question worked fully—

38·196418 ÷ ·214935 to nearest first decimal place
Quotient = 381·96418 ÷ 2·14935.

```
              177·71 |
       ______________|_____
      2·14935)381·96 | 418 ........1
              214·93 | 5
              _______|__
              167·02 | 91  ........2
              150·45 | 45
              _______|_____
               16·57 | 468 ........3
               15·04 | 545
               ______|______
                1·52 | 9230 .......4
                1·50 | 4545
                _____|_______
                 ·02 | 46850 ......5
                 ·02 | 14935
                 ____|_______
                                 Result: 177·7.
```

Note that since main part of divisor is **2** *units*, to secure, say, the *tens* figure of our quotient we divide tens, *i. e.* **16** tens in second stage; or to secure, say, the *tenths* figure of our quotient, we divide *tenths*, *i. e.* **15** tenths in fourth stage. But last figure necessary is hundredths, therefore we need not divide anything less than hundredths in our dividend. Thus in last stage (5), to find *hundredths*, we divide **2** *hundredths* by **2**, taking into account the carrying figures. Therefore in this stage, apart from carrying figures, we require only ·**02**. It follows that in stage (4) we need only **1·52** to ensure both stages (4) and (5) being correct; in stage (3) only **16·57**; in stage (2) **167·02**; in stage (1) **381·96**. In other words, if last figure required in quotient is hundredths, we may draw a line down after **6** hundredths in the dividend and regard it as a barrier past which work need not pass. Had last figure required been thousandths, the barrier would have followed after the third decimal place.

We reach, therefore, this contracted form—

$$
\begin{array}{rr|l}
 & 177{\cdot}72 & \\
2{\cdot}\cancel{1}\cancel{4}\cancel{9}\cancel{3}\cancel{5}) & 381{\cdot}96 & 418 \\
 & 214{\cdot}93 & \\
\hline
 & 167{\cdot}03 & \\
 & 150{\cdot}45 & \\
\hline
 & 16{\cdot}58 & \\
 & 15{\cdot}04 & \\
\hline
 & 1{\cdot}54 & \\
 & 1{\cdot}50 & \\
\hline
 & {\cdot}\ \ 4 & \\
 & 4 &
\end{array}
$$

Result : **177·7.**

(i) No figures required in dividend after hundredths; draw barrier after hundredths.

(ii) **3** hundred ÷ **2** gives **1** hundred. Only **5** figures are required in product; ∴ only **5** are required in divisor; **5** is crossed off except for carrying, the last figure being either **3** or **4**. In other words, divisor is **2·1493.**

(iii) **16** tens ÷ **2** gives approximately **7** tens. Were divisor **2·1493** kept, work would pass beyond barrier. Therefore cross off **3** and have **2·149.** **7** × **3** = **21,** however; carry **2.**

(iv) **16** units ÷ **2** gives approximately **7** units. To keep work to left of barrier, reduce divisor to **2·14.**

(v) **15** tenths ÷ **2** gives approximately **7** tenths. Divisor is now **2·1.**

(vi) **4** hundredths ÷ **2** gives **2** hundredths. Divisor is now **2.**

The rule thus reduces itself to—

(i) Change to standard form, so that divisor is always approximately a number of units less than **10.**

(ii) Draw a line barrier, one place beyond required place, to obtain additional place.

(iii) Keep all work to left side of barrier except for mental consideration of carrying figures.

(iv) In some questions, as in case above, contraction begins before first product is taken away.

In others contraction begins just after first product is taken away.

In others contraction does not begin until two or more products have been taken away.

These divergences are all covered by the rule : " Divide as for full method until work is passing beyond barrier. Then contract divisor one place each time a product is taken away."

N.B.—This method does not ensure the correctness of the additional place taken; *e.g.* in above question we have two hundredths in contracted form instead of one hundredth in full form.

4. Application to Practice. See Chapter VI, pp. 87–89.

5. Application to Compound Interest.

e.g. Find the Compound Interest on £486 7*s.* 6*d.* for $2\frac{1}{2}$ years at $3\frac{1}{4}\%$ per annum.

Answer must be correct to nearest penny, ∴ work two additional places for addition.

£486·375 . .	1st Principal	
14·59125	3% Interest	(*i.e.* × 3 and move two places to right)
1·21594	$\frac{1}{4}\%$ Interest	(*i.e.* 3% ÷ 12 or Principal ÷ 400)
502·18219	2nd Principal	
15·06546	3% Interest	(19 not required, save for carrying)
1·25545	$\frac{1}{4}\%$ Interest	
518·50310	3rd Principal	
5·18503	1% Interest	($\frac{1}{2}$ of $3\frac{1}{4}\% = 1\frac{5}{8}\%$)
2·59251	$\frac{1}{2}\%$ Interest	($\frac{1}{2}$ of 1%)
·64813	$\frac{1}{8}\%$ Interest	($\frac{1}{4}$ of $\frac{1}{2}\%$)
526·92877	Amount after $2\frac{1}{2}$ years	
486·375	1st Principal	
40·55377	Compound Interest for $2\frac{1}{2}$ years.	

Interest = £40·554

= £40 11*s.* 1*d.* to nearest penny.

CHAPTER XI

FACTORS AND MULTIPLES

THIS section of the syllabus in Arithmetic, being of comparatively slight importance and of no undue difficulty, need not long detain us. It may, however, be useful to estimate the value of this work and indicate suitable lines of approach to it.

Probably its greatest value lies in the opportunity it gives children of becoming more familiar with the composition of numbers, and thus of becoming more quick and accurate in shortening calculations by cancellation and factorisation. Special skill in finding the L.C.M. of several numbers is, of course, useful for shortening addition of vulgar fractions, and for solving a few problems involving the finding of a number which contains several others an exact number of times. Special skill in finding the H.C.F. of several numbers is no longer needed to help us to find their L.C.M., and so to deal with them as fractional denominators, because large denominators are now avoided by the use of decimals; thus the usefulness of being able to find a number which will divide exactly into several other numbers is limited to the solution of a few problems demanding this ability. But, apart from this, a certain amount of practice in finding H.C.F. and L.C.M. by factorisation is useful, because it supplies drill in tables from a new point of view, while it makes children quicker to see what cancellation and factorisation are possible in general calculations and also emphasises relationships between numbers.

When we come to consider at what stage we can introduce children to this work so as to make them feel its value, we find little difficulty. As soon as children have begun to add and subtract fractions of any degree of difficulty, they come across a difference of opinion among themselves as to which is the simplest common denominator. Here is a starting-point. The teacher knows a good way of finding a common denominator, and tells the class that it is all a matter of knowing of what a number is composed, suggesting that they drop fractions for a time until they have acquired further skill in dealing with the composition of numbers. Thus we think it wise to take easy addition and subtraction of fractions, as suggested in Chapter IX, p. 126, *before* introducing any detailed study of H.C.F. and L.C.M., and then later to revise the fraction work and to extend it, by using the newly acquired skill in finding L.C.M. So fractions help factors and factors help fractions, while, incidentally, we make our children keen on finding out the composition and relationship of numbers, which will be useful to them in all calculations.

We shall subdivide the ground to be covered into a note on the decomposition of a number into its prime factors; a suggestion for giving meaning to the operations of finding H.C.F.'s and L.C.M.'s; a development of (i) the factorisation rule for H.C.F., (ii) the factorisation rule for L.C.M., (iii) the long rule for H.C.F.; and finally, a suggestion as to other relations between numbers whose study is helpful to calculation. The long rule for H.C.F. has very little value, but as it is still taught in some schools, a few suggestions as to a sensible treatment of it may not come amiss.

Prime Factors

The children set out to gain skill in finding of what a number is composed. They know "factor" in connection with multiplication and division by factors. Hence these stages would be suitable—

(i) Ask for, *e. g.* factors of 96.

$$96 = 12 \times 8$$

We want as many factors as possible: could we get three instead of two?

$$96 = 4 \times 3 \times 8, \text{ or } 6 \times 2 \times 8, \text{ or } 12 \times 4 \times 2.$$

It is wise to make children multiply these together, to convince them that their decomposition is correct.

The rest follows very easily. Find as many factors as possible.

$$96 = 2 \times 2 \times 3 \times 2 \times 2 \times 2 = 2 \times 2 \times 2 \times 2 \times 2 \times 3.$$

96 can now be decomposed no further except by taking unity as a factor.

N.B.—Children may be taught to write this as $2^5 \times 3$. Probably this index notation should be postponed till Standards VI. and VII., as it is a comparatively modern development of symbolism. Certainly it should not be introduced until the need of a short way of writing $2 \times 2 \times 2 \times 2 \times 2$ is realised.

(ii) After exploring a few numbers to their inmost recesses, the terms "prime" and "composite" will be found usefully descriptive, and children can frame suitable definitions.

(iii) When a class advances to finding factors of harder numbers, such as 363 or 162 or 320, various rules for seeing factors can be revised or developed—

e.g. (*a*) Distinguish between odd and even numbers; even numbers can be divided by 2.

(*b*) Numbers are divisible by 4, if the last two figures are divisible by 4; for 4 goes exactly into any number of hundreds.

(*c*) Numbers ending in 5 or 0 are divisible by 5.

(*d*) Numbers are divisible by 3 or 9, if the sum of their digits is divisible by 3 or 9. It is not, however, wise to take this with every class, its usefulness being less than that of (*a*), (*b*), and (*c*), while the reason for it is much harder to see.

Children should be trained to search for factors in an orderly fashion. But a certain fascination which this work has for children makes it easy to teach.

Common Factors and Multiples

The ideas of these and their nomenclature should be developed in connection with measurement of quantity—

e.g. (i) L.C.M.

A. I buy cloth which I intend to cut into pieces 3 yds. long; how much can I buy and have none over?

Multiples of 3 yds.: 3, 6, 9, 12, 30 yds.

B. I buy cloth which I may have to cut into pieces either 3 or 5 yds. long; how much can I buy and provide for both contingencies?

Common Multiples of 3 and 5 yds.: 15, 30, 45, 60 yds. Compare "common" stair: having goods in "common."

C. In question *B*, what is the shortest piece of cloth I can buy?

The *Least Common Multiple* of 3 and 5 yds. is 15 yds.

(ii) H.C.F.

A. I have 42 children in my class and wish to arrange them for drill in equal rows; how many shall I put in a row?

Factors of 42 children: 1, 2, 3, 6, 7 children.

B. I have 48 children in Standard IV., 42 children in Standard V., drilling together, whom I wish to arrange in equal rows; how many shall I put in a row?

Common Factors of 48 and 42 children: 1, 2, 3, 6 children.

C. In question *B*, what is the longest row I can have?

Highest Common Factor of 48 and 42 children is 6 children.

Factorisation Rule for finding H.C.F.

(*a*) Revise the meaning of H.C.F. by taking examples so simple that the answer to them is easily seen without any special working.

(*b*) To lead to discovery of the rule, set such harder examples as: If two pieces of string, 96 in. and 128 in. long, are to be cut up into an exact number of pieces, each of the same length, what is the length of the longest piece which can be obtained?

In order to find out the greatest length of string which can be cut from each, we have to find a number contained exactly in 96 and 128. We must, therefore, know what is *inside* each number, *i. e.* the prime factors of each.

$$96 = 2 \times 48 = 2 \times 2 \times 24 = 2 \times 2 \times 2 \times 2 \times 2 \times 3.$$
$$128 = 2 \times 64 = 2 \times 2 \times 32 = 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2.$$

The factors belonging to both are 2, 2, 2, 2, 2: in other words, the H.C.F. is 32.

Apply this result to the question asked and so prove that it is correct.

A few similar examples worked and fully discussed will establish the rule on a sensible foundation.

(*c*) To fix the new idea and method, give drill work with a variety of numbers and also apply the new skill to the solution of mixed problems.

Factorisation Rule for finding L.C.M.

(*a*) Revise the meaning of L.C.M. by taking examples so simple that the answers to them are easily seen without any special working. In particular, emphasise the fact that the L.C.M. of two numbers prime to each other is their product.

(*b*) To lead to discovery of the rule, set such harder examples as: (i) Find the smallest distance which can be measured exactly by pieces of tape 15 in. and 57 in. long respectively. (ii) What is the smallest sum of money that contains exactly 1*s*. 9*d*., 2*s*. 4*d*., and 3*s*. 6*d*., *i. e.* that contains exactly 21, 28, and 42 pence?

(i) We have to find a number as small as possible to contain exactly 15 and 57. We need, therefore, to know what these numbers themselves contain, *i. e.* we need to know their prime factors.

$$15 = 3 \times 5.$$
$$57 = 3 \times 19.$$

We want a number to contain 15; therefore two of its factors must be 3 and 5. The number must also contain 57; therefore two of its factors are 3 and 19. But there is already a 3 for 15; therefore we need only take 19 as additional factor. Thus the factors of the L.C.M. are 3, 5, and 19.

Or the L.C.M. $= 3 \times 5 \times 19 = 15 \times 19 = 285$.

Apply this result to the question asked and so prove that it is correct.

(ii) We have to find a number as small as possible to contain 21, 28, 42. Therefore we start by finding what are the prime factors of these numbers.

$$21 = 3 \times 7.$$
$$28 = 2 \times 2 \times 7.$$
$$42 = 2 \times 3 \times 7.$$

The L.C.M. must contain 21, therefore it must have 3 and 7 as factors. Since it must contain also 28, it must have two 2's and 7 as factors. To hold, therefore, both 21 and 28 is must have as factors 3, 7, and two 2's. But the L.C.M. has also to contain 42; therefore it must include amongst its factors 2, 3, and 7. It already has two 2's, 3, and 7, so that it is unnecessary to give it any additional factors.

i. e. the L.C.M. $= 2 \times 2 \times 3 \times 7 = 12 \times 7 = 84$.

Test to see whether 84 pence or 7 shillings contains exactly 1*s.* 9d., 2*s.* 4*d.*, and 3*s.* 6*d.*

A few similar examples worked and fully discussed will establish the rule on a sound foundation.

(*c*) If required, lead gradually to the other way of setting work down, *i. e.* by placing the common factors at the side and taking them only once. The above method is, however, preferable.

e. g. (i) L.C.M., as above, of 15 and 57.

$$\begin{array}{r|rr} 3 & 15 & 57 \\ \hline & 5 & 19 \end{array}$$

3 is common factor taken out of both but required in L.C.M. only once.

5 and 19 each belong to one number and appear in L.C.M.

Thus L.C.M. $= 3 \times 5 \times 19$, multiplying together the numbers at the side and on the bottom line, according to the usual rule.

(ii) L.C.M., as above, of 21, 28, and 42.

$$\begin{array}{r|rrr} 7 & 21 & 28 & 42 \\ \hline 3 & 3 & 4 & 6 \\ \hline 2 & 1 & 4 & 2 \\ \hline & 1 & 2 & 1 \end{array}$$

7, 3, 2 are factors common to two or three of the numbers, and therefore appear in L.C.M. only once.

1, 2, 1 are the factors left in each number when as many of the common factors as possible have been taken out.

Thus the L.C.M. $= 7 \times 3 \times 2 \times 1 \times 2 \times 1$,

i. e. $7 \times 3 \times 2 \times 2$, as we obtained before.

(*d*) To fix the new idea and method, give drill work with a variety of numbers and also apply the new skill to the solution of mixed problems.

Long Rule for finding H.C.F.

(*a*) Take a simple concrete example: *e. g.* a room, 40 ft. long and 15 ft. wide, is to have a stencil pattern painted round the tops of its walls. Each side must contain an exact number of patterns; find the size of the largest pattern possible. Illustrate throughout with diagrams and strips of paper or string.

(i) The pattern ends exactly at the corners; can we fix any other points at which it will end exactly? If it is contained exactly in 15 ft., it must end exactly on the long walls 15 and 30 ft. along them; *i. e.* the pattern must be contained exactly in the remaining 10ft.

Or, the H.C.F. of 15 and 40 is the H.C.F. of 10 and 15.

$$\begin{array}{r|r|l} 15 & 40 & 2 \\ & 30 & \\ \cline{2-2} & 10 & \end{array} \qquad\qquad \begin{array}{l} 15)40(2 \\ \quad\;\; 30 \\ \quad\;\; \overline{10} \end{array}$$

(ii) If the pattern is contained exactly in 15 and 10 ft., can we fix any other points on the short walls where it will end exactly? If it is contained exactly in 10 ft., it must end exactly on the short walls 10 ft. along them; *i. e.* the pattern must be contained exactly in the remaining 5 ft.

Or, the H.C.F. of 10 and 15 is the H.C.F. of 10 and 5.

1	15	40	2
	10	30	
	5	10	

15)40(2
30
10)15(1
10
5

(iii) If the pattern is contained exactly in the 10 ft. on the long wall and also in the 5 ft. on the short wall, can we fix any other points on the long wall where it will end exactly? It must end exactly 5 ft. along the 10 ft. and also 10 ft. along the 10 ft. This means, however, that the pattern 5 ft. long ends exactly at the corners of both long and short walls, or 5 ft. is the required length.

The H.C.F. of 10 and 5 is 5.

Therefore the H.C.F. of 15 and 10 is 5.

Therefore the H.C.F. of 40 and 15 is 5.

1	15	40	2
	10	30	
	5	10	2
		10	

15)40(2
30
10)15(1
10
5)10(2
10

(*b*) Repeat this with a variety of simple concrete examples, taking concurrently the reasoning, the illustration by diagrams, paper, or string, and the usual way of setting down the working. Gradually only the working comes to be taken.

(*c*) When the rule has been grasped for small numbers, it can be extended to large numbers without difficulty. But its value is so small that it may well be entirely omitted.

Relationships between Numbers

We give a few suggestions for work which can be taken from time to time in upper standards. Its main value lies in the development of short cuts. When these results are generalised and expressed in symbolic form, they form a useful piece of algebraic work. See Chapter XIX for further suggestions.

(*a*) Take two numbers. Find their L.C.M. and H.C.F. Find product of the two numbers. Find product of their L.C.M. and H.C.F. These two products are alike.

Reason—

Product of the two numbers contains all factors of the numbers.

H.C.F. contains common factors taken once.

L.C.M. contains all factors except the common factors which are taken once instead of twice.

Therefore, H.C.F. × L.C.M. contains all factors of the two numbers.

Thus product of H.C.F. and L.C.M. is the product of the two numbers.

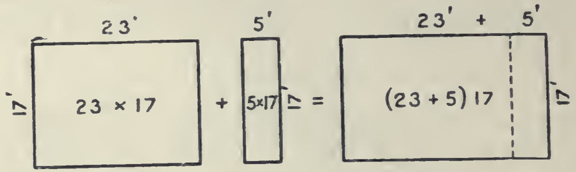
This result does not apply to more than two numbers. The reason for this should be developed. The proof is on similar lines to the above.

(*b*) Take a series of operations involving the same number as factor, *e. g.* $23 \times 17 + 5 \times 17 - 9 \times 17$, and compare with, in this case, $(23 + 5 - 9) \times 17$.

$$\text{Or, } 23 \times 17 - 10 \times 34 = 23 \times 17 - 20 \times 17 = 3 \times 17 = 51.$$

$$\text{Or, } \frac{23 \times 13 + 26 \times 12}{39 \times 7 - 13 \times 6} = \frac{23 \times 13 + 13 \times 24}{13 \times 21 - 13 \times 6} = \frac{47 \times 13}{15 \times 13} = \frac{47}{15}.$$

This can be illustrated by drawing rectangles—



This idea is frequently of use in calculation—

e. g. (i) Find the cost of 37 tables at £1 4*s.* 5*d.* and of 74 pillows at 8*s.* 7*d.*

Cost = 37 × £1 4*s.* 5*d.* + 37 × 17*s.* 2*d.*
= 37 × £2 1*s.* 7*d.*, two multiplication sums being replaced by one.

(ii) Find how much longer the circumference of a plate 12 in. in diameter is than the circumference of a plate 9 in. in diameter.

Circ. of first plate $= \pi \times 12$ in.
Circ. of second plate $= \pi \times 9$ in.
Increase of length $= \pi \times 12 - \pi \times 9$ in.
$= \pi \times 3$ in.
$= 3\frac{1}{7} \times 3$ in.
$= 9\frac{3}{7}$ in.

(*c*) Take two numbers. Square both and subtract the smaller square from the larger. Then find the sum and also the difference of the two numbers. Take the product of this sum and difference. It is the same as the difference between the squares.

e. g. Let 9 and 5 be the numbers: $9^2 = 81$ and $5^2 = 25$; thus $9^2 - 5^2 = 81 - 25 = 56$.
$9 + 5 = 14$, $9 - 5 = 4$; $14 \times 4 = 56$.

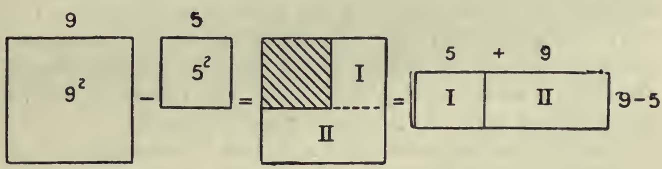
The usefulness of this is apparent in calculation.

e. g. Find the area of a path 3 ft. wide round a circular flower-bed of 24 ft. radius.

Area of path = (area of path and flower-bed) — area of bed.
$= (\pi \times 27^2 - \pi \times 24^2)$ sq. ft.
$= \pi(27^2 - 24^2)$ sq. ft.
$= \pi(27 + 24) \times (27 - 24)$ sq. ft.
$= \pi \times 51 \times 3$ sq. ft.
$= 153\,\pi$ sq. ft.

This also can be illustrated by diagrams—

e. g. $9^2 - 5^2 = (9 + 5)(9 - 5)$.



CHAPTER XII

PROPORTION

One method of calculation which is involved in the solution of the great majority of miscellaneous Arithmetical problems is of such importance that it has to receive special treatment, at latest, in Standard V.; we refer to calculations connected with the idea of proportion. How such a fundamental type of calculation is best taught, at what age it can be begun, which method answers best to the child's natural mental activity, are questions still to some extent without definite replies.

Perhaps the simplest way of approach is to work in grown-up fashion a few typical examples by different methods, and, if possible, decide for ourselves which is the simplest method for adults. Let us take these as our examples—

1. If 65 lb. of a certain tea cost £9 15*s*., what should be paid for 24 lb.?

2. If a supply of rations will last 87 men for 26 days, how long will the same supply last 58 men at the same rate of issue?

3. If a boy 4 years old is 2 ft. high, find the height of the same boy when he is 20.

4. How many men will mow 48 acres of grass in 8 days, if they work at the rate at which 12 men work who mow 36 acres in 5 days?

A. Method of Unity, as practised by adult.

1. 65 lb. of tea cost £$9\frac{3}{4}$.

24 „ „ $£\frac{9\frac{3}{4}}{65}$ (cost of 1 lb.) $\times$ 24.

$$= £\frac{39}{4} \times \frac{1}{65} \times 24$$

$$= £\frac{18}{5}$$

$$= £3\tfrac{3}{5}$$

2. Rations last 87 men for 26 days

„ „ 58 „ 26 days × 87 (for 1 man) ÷ 58 (for 58 men)

$$= \frac{26 \times 87}{58} \text{ days}$$

$$= 39 \text{ days.}$$

3. The boy does not grow at the same rate, therefore boy of 1 year is not 6″ long!

4. 36 acres are mown in 5 days by 12 men,

48 „ „ „ 8 „ $\frac{12}{36} \times 5$ men (1 acre in 1 day) × 48 (48 acres in 1 day) ÷ 8 (48 acres in 8 days)

$$= \frac{12}{36} \times \frac{5 \times 48}{8} \text{ men}$$

$$= 10 \text{ men.}$$

B. Fractional Method, as practised by adult.

1. 65 lb. of tea cost £$9\frac{3}{4}$

24 „ „ £$9\frac{3}{4} \times \frac{24}{65}$ or $\frac{65}{24}$

Answer must be less.

Therefore cost is £$9\frac{3}{4} \times \frac{24}{65}$, or £$\frac{18}{5}$, as before.

2. Rations last 87 men for 26 days,

„ „ 58 „ 26 days $\times \frac{58}{87}$ or $\frac{87}{58}$.

Answer must be greater.

Therefore time is 26 days $\times \frac{87}{58}$ or 39 days, as before.

3. Boy's height at 20 does not bear same relation to his height at 4, as the one age does to the other.

4. 36 acres are mown in 5 days by 12 men
48 „ „ „ 8 „ by
12 men $\times \frac{36}{48}$ or $\frac{48}{36}$ (48 acres instead of 36)
$\times \frac{8}{5}$ or $\frac{5}{8}$ (8 days instead of 5).

In first case answer must be increased, therefore multiply by $\frac{48}{36}$.

In second case answer must be decreased, therefore multiply by $\frac{5}{8}$.

Thus men required are 12 men $\times \frac{48}{36} \times \frac{5}{8}$, or 10 men, as before.

C. PROPORTION.

1. $£x : £9\frac{3}{4} = 24 : 65$ (case of direct proportion)

or $\frac{£x}{£9\frac{3}{4}} = \frac{24}{65}$

$$£x = £9\tfrac{3}{4} \times \frac{24}{65} = £3\tfrac{3}{5}, \text{ as before.}$$

2. x days : 26 days $= 87 : 58$ (case of inverse proportion)

or $\frac{x \text{ days}}{26 \text{ days}} = \frac{87}{58}$

$$x \text{ days} = 26 \text{ days} \times \frac{87}{58} = 39 \text{ days, as before.}$$

3. The boy's height is not proportional to his age.

4. x men : 12 men $= \begin{cases} 48 : 36 \text{ (men directly proportional to acres)} \\ 5 : 8 \text{ (men inversely proportional to days)} \end{cases}$

$$x \text{ men} = 12 \text{ men} \times \frac{48}{36} \times \frac{5}{8}$$
$$= 10 \text{ men, as before.}$$

We shall probably all agree that for the adult the reasoning involved in the method of unity is the most cumbersome. The difference is most marked in a long question, *e. g.* (4) above, but it is obvious that the necessity for reasoning about *one* results in a less simple rule than answering merely the question: "Have I to make my answer greater or less?" As far as ultimate speed goes (see example 4), either the fractional method or the proportional method, which are fundamentally alike, gives much better results. The idea of compounding ratios is probably rather more troublesome and less obvious than the plan of making one definite statement follow another on which it depends. Therefore, from the point of view of the adult, the method of unity comes last; either the fractional method or proportion is to be preferred, the first by those who like a clear statement, the second by those who appreciate the minimum amount of setting down.

From the child's point of view, however, matters are different. No child will naturally use the method of proportion, but children do use both the method of unity and the fractional method, if set simple problems to be worked mentally, their choice depending on the type of question set. We are, therefore, reduced in school to one of three alternatives—

(*a*) We may teach method of unity first and later teach proportion.

(*b*) We may teach method of unity first and later teach the fractional method.

(*c*) We may teach the fractional method from the first and later show proportion only as another way of setting down the work.

Certainly the last, if feasible, will be best, as the method taught from the first is used throughout the course, while in (*a*) and (*b*) the method taught first has to be ousted later by a method fundamentally different, for the sake of giving adults a sufficiently quick and powerful instrument for calculation. With (*c*), then, as our goal, let us try to find a way of approach to this goal which will be natural to children.

Introductory Work in Lower Standards

The idea of proportion is such a fundamental one in life that there is no reason why questions involving the intuitive recognition of it should not be answered by quite little children. Any girl who has dressed a doll, or any boy who has made a model of a ship or an engine, has applied the idea of proportion. Such a question as: "That window is 3 ft. wide and 5 ft. high. I make a drawing of it, making it 6 in. wide; how high should I make it?" is not difficult, and the comparison of the window with right and wrong drawings of it can be used to develop the idea of proportion and the meaning of drawing objects "out of proportion." This is not to say that the child will be able to explain the relation he intuitively recognises, or that he will use the term "proportion"; but it does mean that by "common sense" a child can answer many simple questions involving both direct and inverse proportion. In particular, he applies his knowledge to the making of plans and models and the dressing of dolls and soldiers. "Common sense" covers here the results of experience culminating in an intuitive appreciation of a certain relation.

But, until children can deal with multiplication and division of fractions (that is, about Standard V.), it is wise to restrict all questions asked to certain types, soluble very simply, either by the method of unity or by the fractional method; or, in the case of problems involving harder numerical relations, to demand only an answer to the question: "More or less?" *e. g.*—

> 3 children can go to Keighley and back by train for 9*d*. What will it cost to take 5 children? (Here fare of one child is obvious.)
>
> 6 children can go to Bradford for 2*s*. 9*d*. What will it cost to take 12 or 3 or 18 children? (Cost is obviously twice or half or three times as much.)
>
> A piece of cloth can be cut into three pieces, each 14 yds. long; into how many pieces, each 7 yds. long, can I cut it? (Obviously twice as many.)

A tin of syrup lasts a family of 5 people for 8 days. If they ate at the same rate, would the tin last 7 people a longer or shorter time?

12 umbrellas cost £4 7*s*. 6*d*.; would 17 (or 5) similar umbrellas cost more or less money?

If we restrict ourselves to such work, we shall not meet any of the real difficulties of written proportion questions. It will be found that children's minds respond naturally to the suggestion to ask about *one*; but, to all appearances, equally naturally to the suggestion to look how many times greater or less the answer must be.

Teaching of Written Proportion

This work falls easily into four distinct stages—

A. A development of the recognition of the difference between questions unanswerable by proportion and questions answerable by direct proportion.

A development of a fractional rule for answering questions answerable by direct proportion.

B. An extension of *A* to the difference between questions answerable by direct proportion, questions answerable by inverse proportion, and questions answerable by neither.

A development of a fractional rule for answering questions answerable by inverse proportion, leading to a simple mechanical rule covering both direct and inverse proportion.

C. The extension of such a rule to questions involving changes in three or more related quantities.

D. The comparison of these results with the results achieved by the statement of a proportion and the solution of a simple equation.

Stages *C* and *D* obviously come much later than stages *A* and *B*, which are, indeed, best taken close together, without, of course, undue haste.

Stage A.

(*a*) This stage is best taken by setting questions easily solved mentally, and by correcting these with the class in such a way that the fundamental difference is brought out. As we aim at developing the fractional method, we set mental questions more easily solved by that method than by the method of unity. The following set of questions was given to an average Standard V. class to whom written work in this was new.

1. The diameter of a halfpenny is 1 in.; find the diameter of a threepenny bit.
2. 7 lb. of tea can be bought for 15*s.*; find the cost of 14 lb. of the same tea.
3. A boy, 4 years old, is 2 ft. high; find his height when he is 20.
4. 12 men can build 16 yds. of a wall in a day; how much of the same wall would 3 men build in a day, working at the same rate?
5. 108 bricks were required to make 3 ft. of a certain wall; how many bricks would be required to make 10 ft. of the same wall?

From an analysis of children's attempts at these questions, the following points were brought out—

(i) Some problems are of a kind where the statement helps you to answer the puzzle part (*e.g.* in questions 2, 4, and 5); in some the statement is of no use (*e.g.* in 1 and 3). Other examples of the second type amuse the children, and can be introduced at intervals as catches: *e.g.* 40 potatoes can be boiled in 40 minutes; how long will it take to boil one potato? 2 peacocks waken one man; how many peacocks will be required to waken 6 men? A big gun sends a shell 8 miles in one minute; how long will it take to send it 80 miles?

(ii) Exactly what was done in question 2 was that 15*s.* was multiplied by 2; 2 came from dividing 14 by 7,

$$i.e.\ 7\,\underline{|\,14},\ \text{or}\ \frac{14}{7},\ \text{or}\ 14 \div 7.$$

Children discussed the way of setting down the work.

7 lb. cost 15*s.*

$$14\ \text{,,}\quad \text{,,}\quad 15s. \times \frac{14}{7},\ \text{or}\ 15s. \times 14 \div 7$$

The way of setting down exactly what was done is, of course, important as a help to working harder sums later. Discuss the cost of 21 lb., $3\frac{1}{2}$ lb., $17\frac{1}{2}$ lb., noting that 3 is $\frac{21}{7}$, $\frac{1}{2}$ is $\frac{3\frac{1}{2}}{7}$, $2\frac{1}{2}$ is $\frac{17\frac{1}{2}}{7}$.

(iii) Similar handling of question 4 brought out that 16 was divided by 4 or multiplied by $\frac{1}{4}$, and that $\frac{1}{4}$ came from 3 divided by 12, or the question: "What part is 3 of 12?" Therefore setting down was—

12 men build 16 yds.

3 ,, ,, 16 yds. $\times \frac{3}{12}$ or, 16 yds. $\div \frac{12}{3}$.

The first form was preferred because it was similar to previous case. Compare the work of 24 men, 6, 30, and 72 men.

(iv) Question 5 was attempted in the time only by quicker pupils. Some did it by unity: 3 ft. need 108 bricks, 1 ft. needs 36 bricks, 10 ft. need 360 bricks. But the majority, having just done 2 and 4 by fractional method, thought of 10 as $3\frac{1}{3}$ times 3, saying 9 ft. need 324 bricks, 1 ft. needs 36 bricks, 10 ft. need 360 bricks. They appreciate the speed of the fractional method when led to it from this work.

3 ft. need 108 bricks
10 ,, 108 bricks × ?
9 ft. would need 3 times 108
12 ,, 4 ,, ,,
So 10 ,, $3\frac{1}{3}$,, ,,

It is quicker finally to set down $\frac{10}{3}$ than to think out that this equals $3\frac{1}{3}$, but no point should be forced at this early stage.

Little more can be done in a first lesson.

(*b*) Three or four succeeding lessons are usually sufficient to develop the simple mechanical rule based on a sensible appreciation of the reason for it. The second lesson best takes the form of setting a few similar questions, but in correcting them, analysing them, and definitely passing from the easy to the hard: *e.g.* 6 boys need £2 15*s.* for a week's food; what will be required for 30 boys eating at the same rate?

After answering this as—

$$\begin{array}{l} \text{6 boys need } £2\tfrac{3}{4} \\ 30 \quad \text{,,} \quad \text{,,} \quad £2\tfrac{3}{4} \times \frac{30}{6}, \end{array}$$

which finally works out as £13 15*s.*, children should be asked how much 36, 33, 43 boys would need, and should be led to see that it is easier to set down $\frac{43}{6}$ or $43 \div 6$, instead of first thinking out that this equals $7\frac{1}{6}$.

Before this lesson is over, a great part of the class should be answering correctly by the fractional method questions whose answers are not obvious.

(*c*) In a third lesson it is well to develop the underlying idea more explicitly. This can be done (i) by asking for the difference between the sums they can answer in this way and those they cannot answer, so bringing out the fact that this is a kind of sum where we are told something which helps us to answer something else; (ii) by asking for a fuller description of the new kind of sum they are doing, so leading to the idea that there is some definite relation between the two lines. This can be brought out, *e.g.* in the above question, by asking for the relation between 6 boys and 30 boys and the relation between £2 15*s.* and £13 15*s.*

The nature of this relation can be made clearer by questioning children about making a model, say, of their own sitting-room, where the side of a box 2 ft. long has to serve for a wall which is really 20 ft. long. Finding the correct height and length of the model of the table, the circumference of its legs, the size of the chairs, the position of the table to correspond to its position in the room, all will bring out the fact that the relationship is always one tenth. If wrong measurements are used, the model will not look like the room: it will be "out of proportion." Hence we say the model must be "in proportion." To illustrate the other side of matters, use can be made of Mr. Wells' *Food of the Gods*, a story of food which, when given to infants, caused them to grow into giants! If their height is always twice as great as it should be, children can discuss the building and furnishing for the giants of a sitting-room similar to their own. Everything must be twice as large: anything not twice as large would be out of proportion. Drawings of objects in the room out of proportion and in proportion can be examined. The point to be emphasised finally is, of course, that each quantity in the second line must be a certain part of, or a certain number of times, the corresponding quantity in the first line.

(*d*) From this onwards, nothing remains save to perfect the mechanical rule and to extend it to compound quantities.

e.g. I require 1 hr. 50 min. to walk $5\frac{1}{2}$ miles; at the same rate, how far can I walk in 4 hr. 35 min.?

$$1\tfrac{5}{6} \text{ hr. for } 5\tfrac{1}{2} \text{ miles}$$

$$4\tfrac{7}{12} \quad ,, \quad 5\tfrac{1}{2} \text{ miles} \times \frac{4\frac{7}{12}}{1\frac{5}{6}}$$

$$= 5\tfrac{1}{2} \text{ miles} \times 4\tfrac{7}{12} \div 1\tfrac{5}{6}$$

$$= \frac{\overset{1}{\cancel{11}}}{2} \text{ miles} \times \frac{55}{\underset{2}{\cancel{12}}} \times \frac{\cancel{6}}{\underset{1}{\cancel{11}}}$$

$$= \tfrac{55}{4} \text{ miles or } 13\tfrac{3}{4} \text{ miles.}$$

In such examples the superiority of this fractional method over the method of unity immediately appears.

Stage B.

(*a*) This stage is again best introduced by setting a list of questions soluble mentally, including examples illustrating inverse proportion. The following were set to an ordinary large Standard V. class which had been at *Stage A* for rather less than a fortnight—

1. A horse can pull a load of 1 ton 20 miles in a day: how far can it pull a load of 20 tons in a day?
2. 3 pinafores can be made from $4\frac{1}{2}$ yds. of print: how many yards shall I require to make 9 similar pinafores?
3. 4 porters can empty a luggage van in 4 minutes: how long would 8 porters take to empty the same luggage van if they work at the same rate?

N

4. A piece of cloth can be divided into 6 lengths of 7 yds.: into how many lengths of 21 yds. can it be cut?

5. A square of side 5 ft. has an area of 25 sq. ft.: what is the area of a square of side of 10 ft.?

6. I can buy 12 copies of a certain book for £4 10*s.*: at the same rate, what must I pay for 9 copies of the same book?

The children on the whole did these well. Half the class had four right and a few had all right. The results were taken rather rapidly, and the class then divided the questions set into three types and described each type. One pair, 1 and 5, was such that the given statement had no effect on the answer required; in the remaining four questions it did affect the answer. In 2 and 6 the relation was such that when one quantity increased (or decreased) the other quantity increased (or decreased) in such a way that the same number of times (or the same part) was taken. This was their old friend, direct proportion.

But in 3 and 4 children noted that things were "topsy-turvy"! The porter question was dramatised and a table made—

	Porters.	Minutes.
	4	4
	8	2
	16	1
	32	$\frac{1}{2}$
	64	$\frac{1}{4}$
	2	8
	1	16
(a very weak man)	$\frac{1}{2}$	32

This soon brought out the fact that when one was increased (or decreased) the other was decreased (or increased). Halving or quartering seemed to be the topsy-turvy way of taking 2 or 4 times something. The question was then written down and compared with direct proportion work—

4 porters empty van in 4 min.
8 " " " 4 min. $\div$ 2

2 was explained as $\frac{8}{4}$

4 min. $\div \frac{8}{4}$ was compared with the direct proportion sums where $\div$ would have been $\times$. If we wished to use $\times$ instead of $\div$, children said it would be 4 min. $\times \frac{4}{8}$. In fact, the lesson ended with the idea of inverse reached very hazily, as being a topsy-turvy way of things—decreasing instead of increasing, division instead of multiplication, $\frac{4}{8}$ instead of $\frac{8}{4}$. Children were asked what "inverse" or "topsy-turvy" addition would be: they said "subtraction," showing that the idea of inverse was to some extent realised.

(*b*) The progress of this same class in a second lesson was amazing, the rule which covers both direct and inverse proportion being reached, to the teacher's great surprise, a result which convinced her of the superiority of the fractional method over the method of unity, even in Standard V.

Following the previous plans, six questions were put on the blackboard, which the children were asked to work, mentally if possible, but with the warning that probably the last three would be the better for a full statement.

1. A recipe for making pies states: "Take $\frac{1}{2}$ lb. of butter with every $1\frac{1}{4}$ lb. of flour." What weight of butter will be required for $3\frac{3}{4}$ lb. of flour?

2. A cricketer makes 24 runs in the first half-hour of his innings. How many runs will he make if his innings lasts $1\frac{1}{4}$ hr.?

3. A plot of ground was 6 ft. wide and 12 ft. long. Another plot, of the same area, was 2 ft. wide: how long was it?

4. In making plum jam a woman adds $\frac{3}{4}$ lb. of sugar to $1\frac{1}{4}$ lb. of fruit. What weight of sugar will she require when the weight of the plums is $12\frac{1}{2}$ lb.?

5. A shop was lit by 16 burners which consumed a certain amount of gas in 60 minutes: if the number of the burners is increased to 24, how long will the same amount of gas last?

6. A large pile of envelopes can be addressed by a staff of eight clerks in $12\frac{1}{2}$ hours. If only 5 clerks had been working at the same average rate, how long would they have taken?

The children spent fifteen or twenty minutes over these, with varied results. The first four were, on the whole, answered well, though some children made mistakes in each of them. Questions 5 and 6 were answered correctly by about six children. Some of these reached their correct answer after first working the questions by direct proportion; finding their answers ridiculous, they inverted their fractional multiplier without any advice from their teacher!

The first three were then taken briefly as revision of the three different types. Question 3 was taken in detail with diagrams, and the following points were made—

Area remains unaltered, therefore

when breadth is got by multiplying 6 by $\frac{1}{2}$, length is got by multiplying 12 by 2;

when breadth is got by multiplying 6 by 3, length is got by multiplying 12 by $\frac{1}{3}$;

when breadth is got by multiplying 6 by $\frac{2}{3}$, length is got by multiplying 12 by $\frac{3}{2}$;

when breadth is got by multiplying 6 by $\frac{3}{2}$, length is got by multiplying 12 by $\frac{2}{3}$.

Thus the inverse idea was developed more fully.

Question 4 was written out. In working it, the teacher wrote $\times \frac{1\frac{1}{4}}{12\frac{1}{2}}$, asking class to explain to her why this was wrong. She was promptly told that she had made her answer less instead of greater, by multiplying by the smaller number and dividing by the larger! She then asked whether she should multiply by $\frac{5}{11}$ or $\frac{11}{5}$ to make an answer less: the whole class seemed very much at their ease in this matter.

Question 5 was next tackled.

16 burners used the gas in 60 min.			
32	,,	,,	? min.

was asked as an easy introduction. Most of the class gave 30 min., though a few gave 120 min., and had to think of using the jets of a cooking-stove to see their mistake. This was then set down as 60 min. $\div \frac{32}{16}$, or 60 min. $\times \frac{16}{32}$. Class noted that 32 burners were twice 16 burners, but 30 min. were half of 60 min. Teacher next asked—

16 burners used the gas in 60 min.
24 „ „ „ 60 min. × ?

Children at once said $\frac{16}{24}$! Teacher asked, "Why?" "Because answer must be less." This was worked out as 60 min. $\times \frac{16}{24}$ and the answer given as 40 min. But teacher was doubtful if class were sure that fraction was correct, so asked them to find out how many times 16 burners were 24 burners, and what part of 60 min. was 40 min. When children saw their answers as $\frac{3}{2}$ and $\frac{2}{3}$, they smiled! Teacher asked, "If one had been $\frac{11}{8}$, what would the other have been?" She was pleasantly surprised at the children's quick appreciation of the inverse idea.

Question 6 was taken similarly. Class suggested 4 clerks as easier than 5 clerks, and, after working, noted the relationships $\frac{1}{2}$ and 2. Five clerks were then dealt with without difficulty; children had also to work out the relationship again, and hailed $\frac{8}{5}$ and $\frac{5}{8}$ with pleasure! In answering this question, the children obviously asked themselves only the question, "Greater or less?"

The teacher concluded that nothing could be gained by further delay in introducing children to mixed inverse and direct proportion questions, if these were worked by the fractional method, and the form of the fractional multiplier were decided by simply asking, "Less or more?" It would, of course, be wise occasionally to compare the answer with the statement, noting equivalent fractions in the case of direct proportion and inverse fractions in the case of inverse proportion. Clearly the child has here reached the idea of ratio and of equivalent ratios, although the term "ratio" and the usual notation for ratio is best deferred till later, when it can be joined to work in Geometry and Algebra: *e.g.* to the relation between the circumference and diameter of a circle, to the drawing of similar triangles and polygons, and to the solution of simple equations.

Stage C.

When three or more relations are involved in a question, the children quickly learn how to deal with them, one at a time. For instance, in "How many men will be needed to mow 48 acres of grass in 8 days, if they work at the same rate at which 12 men work who mow 36 acres in 5 days?" teach them—

(i) To determine the nature of the answer, men or days or acres.

(ii) To make a statement of what is given in such a way that the unknown element comes, for convenience, at the end of the sentence; *e.g.* 36 acres are mown in 5 days by 12 men.

(iii) To write down a puzzle line to correspond with the statement line; *e.g.* 48 acres are mown in 8 days by ?

(iv) To alter the original number of men to suit each change of circumstances;

e.g. 36 acres are mown in 5 days by 12 men
48 ,, ,, ,, 8 ,, 12 men × ?

When acres are altered from 36 to 48, men must be altered from 12 to $12 \times \frac{48}{36}$.

When days are altered from 5 to 8, men must be altered from $12 \times \frac{48}{36}$ to $12 \times \frac{48}{36} \times \frac{5}{8}$.

Thus number of men required is $12 \times \frac{48}{36} \times \frac{5}{8}$, which works out as **10**.

(v) To ask what question 12 men $\times \frac{48}{36}$ answers, what question 12 men $\times \frac{5}{8}$ answers.

After a few questions have been thoroughly dealt with on these lines, rapid progress becomes possible.

Stage D.

As stated at the end of section *B*, all that is necessary now is to introduce the term "ratio," along with the ideas of equivalent and inverse ratios. Children represent the unknown number by x, say. They write down—

$$x \text{ lb.} : 4 \text{ lb.} = £5 : £8$$

$$\text{or, } \frac{x}{4} = \frac{5}{8}, \text{ as they already know.}$$

Compare with similar triangles, shadows, drawings in proportion. The remaining work is simply the solution of a simple equation—

$$\text{Multiply both sides by } 4,\ x = \frac{5}{8} \times 4 = \frac{5}{2} = 2\tfrac{1}{2}.$$

We should note that there is no need to trouble children over the compounding of ratios, since there is here no question of ousting the fractional method of solving any proportion problem. We are simply developing more fully the nature of direct and inverse proportion, in connection with a definition and examples of ratio, Geometrical examples of proportion, the solution of simple equations and the drawing of graphs.

The point of view of the above chapter may have been expressed too dogmatically for some people's taste! But we have convinced ourselves from experience that, if only we know *how* to teach the fractional method as well as we know how to teach the method of unity, its results will far outshine the old results. We are satisfied that the question of unity *v.* fractional is not a question of the child's ability to understand both, but a question of our own grasp of the relationships called direct and inverse proportion.

CHAPTER XIII

PERCENTAGES

At the beginning of our study of Arithmetical operations we noted how mankind was forced to count and to measure because of the limits imposed by the universe. We have seen how, in measuring quantities in order to achieve some desired end, man learnt to build up new wholes from parts and to break up wholes into new parts, as in addition and subtraction; to measure more exactly by using a simple instead of a derived unit, as in multiplication; to measure the same quantity by a new unit, as in division; to choose certain standard units for the measurement of typical quantities, such as money and length, and to develop fixed connections between these units. We then passed on to study more closely how he dealt with the measurement of parts or fractions of some defined whole, either using the idea and notation of vulgar fractions where any part of the whole may serve as a new unit; or using the idea and notation of decimal fractions where only tenths, hundredths, thousandths, etc., of the original whole may serve as new units. We have noted, in the preceding chapter, that an appreciation of the fractional relationship between connected quantities gives us power to extend knowledge of one set of facts to a great number of correlated facts when the relation involved is direct or inverse proportion. The use of percentages is the only topic of fundamental importance in Arithmetic still untouched, the remainder of the work being applications of Arithmetic either to money problems which arise for us as members of a great community, or to elementary Geometry and Algebra. What, then, is the place of percentages in the above outline of our measuring activities?

Percentages simply afford a third method of measuring fractions of a whole, a method to be set by the side of vulgar and decimal fractions. A percentage is simply a fraction, written with a new notation. We may write the same quantity as £$\frac{23}{100}$ or £·23 or 23% of £1. As we have seen, the vulgar fraction is most suitable for representing simple parts of a whole, or harder parts when it is not necessary to combine or compare these: *e. g.* we deal cheerfully in any way required with £$\frac{5}{8}$ and £$\frac{7}{12}$, but we hesitate to embark on certain operations involving £$\frac{25}{81}$ and £$\frac{73}{124}$. We have noted that in cases where such awkward fractional quantities have to be compared or combined, it is simpler to use the decimal fraction and express these as a sum of tenths, hundredths, thousandths, etc., to any required degree of accuracy. In percentages we take any fractional quantity and express it as a certain number of *hundredths*; *e. g.* £$\frac{5}{8}$ is 62½ hundredths of £1, £$\frac{25}{81}$ is 30$\frac{70}{81}$ or 30·9 hundredths of £1, hundredths usually being written % (N.B.—still a line and two 0's !). By expressing fractions as percentages we have the same ease in comparing them that we had by expressing them as decimals; in fact, the two processes are almost alike, for 62½ hundredths is simply ·625 and 30·9 % is ·309. But we have one additional reason for introducing the percentage notation.

An example will explain this. Standard II., with 50 children on the roll, has 45 present; Standard III., with 56 children, has 49 present; Standard IV., with 40 children, has 38 present. Which class deserves the shield for good attendance ? Here the measurement necessary is a comparison of the relation of the number present to the whole class in each case; the difficulty in comparing these three relations is due to the necessity of comparing parts of unequal wholes. Vulgar fractions simply ask us to decide which is greatest: $\frac{45}{50}$ or $\frac{49}{56}$ or $\frac{38}{40}$? Decimal fractions are much better, giving us ·9, ·875 and ·95 of the three classes respectively as present, with obviously ·95 as measuring the best attendance. Had the fraction been $\frac{42}{49}$ or $\cdot\dot{8}5714\dot{2}$, the decimal method would have been more unwieldy. What is required is really an *approximate* decimal method, which we call a *percentage*, i. e. measurement *per cent.* or out of a hundred, or measurement in hundredths. This gives, in the above example, 90, 87½ and 95 out of a hundred, or 90, 87½, 95 hundredths, or 90%, 87½%, and 95%. $\frac{42}{49}$ would thus be 85·7 %. Here we have a way of measuring the relations of parts to their wholes which gives an easy method of comparing parts of unequal wholes. This kind of comparison is continually necessary; *e.g.* which year gives the best interest in the five years required for the maturing of a War Savings Certificate if 15*s*. 6*d*. amounts—

in 1 year to 15*s*. 9*d*.
in 2 years to 16*s*. 9*d*.
in 3 ,, to 17*s*. 9*d*.
in 4 ,, to 18*s*. 9*d*.
in 5 ,, to £1 ?

Each year the interest is the interest on a different principal and so is best expressed as a percentage of principal. The five percentages can be compared at a glance.

$$\frac{3d.}{15s.\ 6d.} = 1{\cdot}6\ \%$$
$$\frac{1s.}{15s.\ 9d.} = 6{\cdot}3\ \%$$
$$\frac{1s.}{16s.\ 9d.} = 6{\cdot}0\ \%$$
$$\frac{1s.}{17s.\ 9d.} = 5{\cdot}6\ \%$$
$$\frac{1s.\ 3d.}{18s.\ 9d.} \doteq 6{\cdot}7\ \%$$

To sum up briefly, percentages simply afford a third way of measuring parts of a whole, and have a notation closely akin to decimal notation. They have a peculiar usefulness in enabling us to compare the relations of parts to their wholes when the wholes are of unequal magnitude. In practice the percentage notation has proved so admirable that ability to think in terms of percentages is a necessity if one is to read with intelligence modern newspapers, magazines and books. With these ideas in our minds, we now give a brief outline of suitable steps by which our children may be taught percentages.

1. *Introductory Problems to illustrate the Use of Percentages.*

Set a question bearing on attendance or on marks (*e. g.* on which day did I do best work, when I had 8 out of 10, or 12 out of 15, or 30 out of 35?) or on any other topic of interest to children. Such a question should make our pupils feel the need of learning something new and should lead to an introduction of the name and notation of percentages.

Per cent.: out of 100: %.

2. *Squared Paper Illustrations of the Connection of Percentages with Vulgar and Decimal Fractions.*

Having vaguely introduced the new work and established a desire for further acquaintance, we do well to fall back for a time on an appeal to the eyes by using squared paper.

The whole may be called a square inch or £1, or 1 hour; variety is all to the good. As in work with decimals (p. 136) 50%, 25%, $33\frac{1}{3}$%, $12\frac{1}{2}$%, 5%, of the whole can be shaded in, compared with ·5, ·25, $\cdot 33\dot{3}$, ·125, ·05, and with $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{3}$, $\frac{1}{8}$, $\frac{1}{20}$ of the whole. Such a question as, "I spend 50% of a day in sleep, 23% of the day in school; what percentage of the day have I left for other things?" can be easily answered from a diagram. A very little of this work will suffice if decimal notation has previously been well taught.

3. *Making and Use of a Table of Common Percentages.*

From the above diagrams and from simple reasoning a percentage table for $\frac{1}{2}$'s, $\frac{1}{4}$'s, $\frac{1}{8}$'s, $\frac{1}{3}$'s, $\frac{1}{6}$'s, $\frac{1}{5}$'s, $\frac{1}{10}$'s, $\frac{1}{20}$'s, should be drawn up. An immense variety of easy problems soluble mentally either exactly or approximately from this table, should be taken, so that children may gain facility in the use of percentages: *e. g.* "$33\frac{1}{3}$% of this class of girls are wearing navy serge frocks; there are 12 girls with these frocks. How many girls are there in the class?" or "I had 1*s.* and spent $7\frac{1}{2}d.$ What percentage of my money was left?"

4. *Development of Rule for finding the Value of a Percentage of a Given Whole Quantity.*

This work children can easily tackle for themselves, finding out, for example, that—

$$\begin{aligned} 22\% \text{ of } 5 \text{ lb. } 5 \text{ oz.} &= \frac{22}{\underset{20}{\cancel{100}}} \text{ of } \overset{17}{\cancel{85}} \text{ oz.} \\ &= \frac{11 \times 17}{10} \text{ oz.} \\ &= 18{\cdot}7 \text{ oz.} \end{aligned}$$

They should, however, have special practice in finding 5% and $2\frac{1}{2}$% of any sum of money, as ability to do this mentally is often useful.

(*a*) $5\% = \frac{1}{20}$. On £1 we take 1*s.*; on 10*s.*, 6*d.*;
$\therefore$ 5% of £14 10*s.* = 14*s.* 6*d.*

$2\frac{1}{2}\% = \frac{1}{40}$. On £1 we take 6*d.*; on 10*s.*, 3*d.*;
$\therefore$ $2\frac{1}{2}$% of £3 5*s.* = 1*s.* 6*d.* + $1\frac{1}{2}$*d.* = 1*s.* $7\frac{1}{2}$*d.*

(*b*) $5\% = \frac{1}{20}$. On every 20 pence take 1*d.*
$\therefore$ 5% of £14 17*s.* 4*d.* = 14*s.* 6*d.* + 5% of 88*d.* = 14*s.* 10*d*, neglecting fractions of 1*d.*

$2\frac{1}{2}\% = \frac{1}{40}$. On every 40*d.* take 1*d.*
$\therefore$ $2\frac{1}{2}$% of £3 6*s.* 2*d.* = 1*s.* 6*d.* + $2\frac{1}{2}$% of 74*d.*
= 1*s.* 7*d.*, neglecting fractions of 1*d.*

5. *Development of Rule for expressing Part of a Quantity as a Percentage of the Whole Quantity.*

This can be done in two ways.

e. g. I got 11 marks out of 24; how much is that per cent.?

(*a*) Out of total of 24, I got 11 marks.

$$\begin{aligned} \therefore \quad " \quad " \quad 100, \; " \quad &\frac{11 \times \overset{25}{\cancel{100}}}{\underset{6}{\cancel{24}}} \text{ marks.} \\ &= \tfrac{275}{6} \text{ marks.} \\ &= 45\tfrac{5}{6} \text{ marks.} \end{aligned}$$

Result is $45\frac{5}{6}$%.

(*b*) I got $\frac{11}{24}$ of the total marks.

$$\begin{aligned} \therefore \text{ I should have } \tfrac{11}{24} \text{ of } 100\,\% & \\ &= \tfrac{275}{6}\,\% \\ &= 45\tfrac{5}{6}\,\%. \end{aligned}$$

6. *General Problems.*

These are, of course, of varying degrees of difficulty and will be spread over the remaining years of the child's school life. A few of the more important money problems will be treated in the following chapter.

7. *Illustrations from Other Subjects.*

If possible, use should be made of relations between quantities which emerge, say, in Geography or Hygiene or Housecraft, by reducing such relations to percentages and representing the results by means of a chart or the more abstract graph. The type of work taken depends on the work of the class in other subjects, but the teacher should, if possible at all, secure some such illustration in order to make the practical value of percentage work more obvious to his pupils.

CHAPTER XIV

COMMERCIAL AND SOCIAL APPLICATIONS OF ARITHMETIC

UNDER the above heading can be grouped a great variety of topics, on which much time is often spent: we refer to Profit and Loss, Discount, Interest, Stocks and Shares, Rates, Taxes, Bankruptcy. We here summarise the ground to be covered as follows—

(i) Problems on Buying and Selling. (To cover profit, loss, and discount.)
(ii) Problems on Investment of Money. (To cover interest, stocks and shares.)
(iii) Problems on Citizenship. (To cover rates and taxes.)
(iv) Problems on Insurance (*e. g.* life, burial, unemployment, sickness, fire, accident).
(v) Problems on Bankruptcy.

In other words, as members of a community, we have certain problems which do not arise for individuals. We buy and sell; we have "spare cash" to dispose of; we are citizens of towns, counties, and a nation; we are all liable to many risks but do not all face these risks; at times our financial position is unsound. If our children are to be fitted to take their place in the community they must receive some training in these matters.

Regarding this from another standpoint, we may dwell on certain points of method which are implicit in the above statements. First, we teach this work to make our children better members of the community: we must, therefore, so select and impart the knowledge given that this end shall be attained. To do this, it is of essential importance to know the facts for ourselves, and as these facts alter from year to year, especially in times of national crisis, we shall have to study newspapers and haunt post offices, town and county buildings and inland revenue establishments, until we acquire the necessary information. We must also know the social conditions of our pupils so that we can select such illustrations as may be of most use to the future well-being of individuals and of the community.

Next we note that the Arithmetic involved is in no case new work: it is simply an application of the old to new problems which give it a fresh setting. Children have little difficulty in doing sensible work, if the new situation is made real to them, though, of course, they require time to gain the necessary practice. To make the new situation real should not be hard, as in each case it is a vital part of life. If we make as many points of contact as possible with the children's experience outside school; if along with facility in solving problems we also develop the large fundamental ideas; if, as far as possible, we make opportunities for them to dramatise inside school what the grown-ups do outside; then we shall find that the new situations have become so vividly realised by our pupils that they readily apply to the solution of the new problems their previous knowledge of Arithmetic.

Last, we note that we are preparing our pupils to become members of the community, not, in general, specifically trained business men and women. Therefore this work should be taken on broad lines and many things which the specialist must later acquire should be omitted. If this be done, the vital ideas are not buried under a mass of petty detail, but become more potent in their influence on the changing problems which confront us in these tumultuous times.

In dealing briefly with each of these subdivisions, we shall merely indicate possible starting-points, emphasise what seem to us the vital ideas underlying each topic, and make a few suggestions as to suitable illustrations garnered from the children's experience outside school. Here, if anywhere, each must work out the topics for himself in his peculiar circumstances. What follows is to be regarded merely as suggestive, though, for brevity, it is expressed in a somewhat dogmatic form.

Buying and Selling

The idea which seems most in need of emphasis is the value to the community of wholesale and retail dealers. The commodities we want (*e. g.* for food and clothing) must come from many distant places: no longer can each find them for himself. This, of course, correlates with Geography and should be worked out in some detail; it also links itself closely with History, in a contrast of present food and clothing with those of more primitive times. How have we solved this problem of bringing the required commodities to our doors? The wholesale and retail dealers do this, the wholesale dealers buying large quantities of a few things in distant places and bringing them to large depots in the district, while the retail dealers, with their help, are able to buy smaller quantities of a great variety of goods and sell them to us almost at our doors. In certain cases (*e. g.* bakers, butchers and chemists) productive labour, in addition, is given us in retail establishments. This should be worked out in detail. It is all to the good if the underlying schism between co-operation (as in co-operative stores) and competition (as between rival tradesmen) can be brought out and criticised by the fresh and comparatively unselfish eyes of children. Their attitude may surprise us. The essentially unchristian idea of competition does not appeal to the childlike spirit. But here each must act according to the faith that is in him.

How, then, are these wholesale and retail dealers to be paid for their labours on our behalf? Different plans are adopted in co-operative stores, in the ordinary shop, and in great business organisations like Lipton's or the Maypole, where the owners of the enterprise are a company, not one or more individuals. Here we can dramatise the ordinary problem by having a child sell some commodity desired by his companions, pencils or buns or sweets. The advantage of being able to get these things in *school* is clear to the children: the seller confers benefit on them. How is he to be paid? The price he himself paid for the goods can be discussed and a method of deciding at what price he should sell them in order to earn his reward. Children see that to add as profit a certain fraction of the cost price is fairer than to add a certain fixed sum to each article. The advantage of defining this fraction as a percentage is obvious. If work is taken on these lines our pupils are not likely to reckon the percentage gain or loss on the selling price, as that does not answer the requirements of the real situation.

Interesting problems can be worked if children bring information from parents about cost and selling prices of articles they actually sell. Comparison of prices in different districts is of use. Discount for cash can be introduced, and the advantages and disadvantages of cash payments discussed. Our pupils can use their private knowledge to make up questions for their companions to solve. If these are worked along with some of the usual type of book-work question, the subject should be sufficiently developed for most practical purposes.

Investment of Money

The most important idea here is that money grows only by being used in some form of productive work. Interest is not made by magic, but by work. The far-reaching significance of this must be indicated.

We note, first, that this statement applies to all ways of getting interest on money. Mary has 1*s.* to spare, but lacks either time or ability to use it productively; Helen has no money to spare, but has both time and ability to make useful and attractive hair-tidies. She asks Mary for 1*s.*, buys material, makes her articles in a fortnight's spare time and sells them for 2*s.* No child will admit that Helen should keep the extra shilling and return to Mary only one shilling: every child is sure that Mary should have some reward for the loan of her money, though opinions vary widely as to a suitable sum. Children will probably suggest a penny or $\frac{1}{2}$*d.*, halving or doubling this if either the time or the amount lent is halved or doubled. This simple example contains in a nutshell the theory of interest.

A man or a small group of men have spare thousands of pounds. With this they build a mill and introduce into it the most modern machinery for weaving, say, silk goods. They engage employees, buy the raw product, and sell the finished article at a price greater than the price of the raw material together with the employees' wages. The profit they thus make is the interest on their spare thousands of pounds now converted into mill and machinery. Without work there can be no interest. Or, instead of themselves planning and overseeing the work, they may hand the money over to some other group of men running a railway, say, or mining coal, or providing entertainment for the public by a picture palace. This second group of men do the work, but the first group cannot have interest on their money unless the second group do some work. The original group may also put their money into a bank and gain interest in that way. But no bank makes interest by magic: it has men skilled in choosing good working concerns to which to lend the money it has received, skilled, that is, in choosing good investments.

Two other examples must suffice. Recently the nation was investing a large part of its spare cash in War Savings Certificates or in War Loans. Does the statement that interest grows by work apply here? The money was used to provide men, guns, ammunition and all the subsidiary services necessary to carry on a war. On the surface, the work done appears to be destructive, not productive: certainly money used in such a way does not make more money, and the interest paid to individuals has to be provided by the nation from its resources of wealth. But no nation would continue to carry on war unless it believed that, although the work was destructive in the material sense and for the immediate present, yet it would be either materially productive in the remote future, or, sooner or later, morally and spiritually productive. Opinions differ on these matters, but unless the majority of a nation believed this, that nation would cease to wage war.

To descend from the great to the small, does a pawnbroker gain interest on the money he lends because it is used productively? There is no difference of principle here, the difference being merely that the borrower in this case is so little trusted that the lender demands security in goods at least equal in value to the sum of money lent. Until the borrower has spent in a productive way this money, using it to provide food to enable him to do wage-earning work, he will not be able to repay the loan and redeem his goods.

We see, then, that underneath the surface life of the community flows a continuous current of money. To deal with the immense variety of financial problems solved and unsolved by mankind is far beyond the scope of this book. The aim now is to show in clear outline the main features of the situation, for it is of the utmost importance to the well-being of the nation that one and all should learn to think clearly, wisely and unselfishly about the financial problems of the State. Before August 1914 we came too close to civil war between Labour and Capital to view with equanimity any return to similar conditions. The teacher's problem is how to give children the necessary guidance in these matters.

The first great fact is, undoubtedly, that Labour needs Capital. Without Mary's shilling, the mill-owners' thousands, the nation's savings and the pawnbroker's loan, Helen, the silk-weavers, the military and the penniless man might be most willing to work and have the greatest latent abilities, but work they could not. Thus Capital gets its power over Labour.

The next great fact is that Capital has much more choice in the matter. If Capital wishes to gain interest, then it must use Labour; but if it does not care about gaining more money, it can do as it pleases. Children readily give instances of what is done with "spare cash" by the prosperous members of all classes of society. They bring forward definite examples of luxurious spending, of burying talents in a napkin, of the philanthropic use of money, of investment in concerns both beneficial and harmful to the community. They are quick too in expressing their judgments on these matters.

It will be noted from the above illustrations that many members of the nation represent both Labour and Capital; it is to the good of the nation that as many as possible should represent both. As soon as a man earns more than enough for healthy living for himself and his dependents, he is a capitalist who can choose what he will do with his spare money. But on the one side are the idle, be their incomes great or small, those who, being able to work either with hands or intellect, will not. On the other side are the very poor, who, however willing they may be to work, can never reach the margin where they have enough income for healthy life. The existence of these two extremes is, of course, one cause of the embittered relations between Labour and Capital.

The other chief cause of bitterness is more purely an Arithmetical one. Every one realises that those who lend money should receive some return for it; the question as to what return they shall receive is a thorny one. Children will agree that the amount of the interest should be proportional to the time and to the amount of the principal, but on the question of a suitable rate of interest great divergences of opinion exist. Children's sympathies, strange to say, are at first with the capitalists, probably because at this stage they have more often little savings than any definite wages! Halving the gain is often suggested: not many grown-ups would demand 50% interest! Questions of security, time and livelihood must here be considered.

When we come to the details of investments, we do well to start as closely as possible to the children's experience. Many schools run Penny Savings Banks; the working of these should be understood and questions set on the interest gained. Here can be developed the idea of expressing interest as a percentage. We must deal, too, with the Post Office Savings Bank, making use of any forms which we can obtain from the Post Office, *e. g.* a form for 12 penny stamps to let children save pence until they can put a shilling into the Savings Bank. Problems should be sensible and as far as possible dealt with by short methods. Most teachers know that $2\frac{1}{2}$% per annum is 6*d.* per £1 per annum, or $\frac{1}{4}$*d.* per £1 per month: money has to lie in the bank for a month before interest is allowed on it, and interest is calculated only for a whole number of calendar months. Real bank books should be produced and the interest on them calculated. War Savings Associations are run temporarily in some schools: problems on interest arise from these. Children should work out the rates of interest charged by pawnbrokers and moneylenders; some constructive work on better ways of dealing with financial emergencies can in certain schools do considerable good. Pupils should, in better districts, find out how much money they must save to secure a weekly income of from 10*s*. to 20*s*.

As far as the Arithmetical processes go, a variety of methods is possible. In the beginning it is probably best to use the fractional method for proportion.

e. g. (i) Find the Simple Interest on £436 5*s*. for $2\frac{1}{2}$ years at 4% per annum.

S. I. on £100 for 1 year is £4.

$$\text{S. I. on £}436\tfrac{1}{4}\text{ for }2\tfrac{1}{2}\text{ years is £}4 \times \frac{436\frac{1}{4}}{100} \times 2\tfrac{1}{2}.$$

(ii) Find the rate per cent. per annum if the interest on 1*s*. for 1 week is 1*d*.

S. I. on 1*s*. for one week is 1*d*.

S. I. on 100*s*. for 52 weeks is $\frac{1}{12}s. \times 100 \times 52$.

$$\text{or } \frac{1300}{3}s.$$

$$\text{or } 433\tfrac{1}{3}s.$$

Rate is $433\frac{1}{3}$ % per annum!

(iii) Find the S. I. on £P for T years at R% per annum. (See Chapter XIX.)

S. I. on £100 for 1 year is £R.

$$\text{S. I. on £P for T years is £R} \times \frac{P}{100} \times \frac{T}{1}$$

$$\text{Or } I = \frac{R\ P\ T}{100}.$$

From this work children can advance to the use of the formula, if that be desired, but in many schools such work is of comparatively little importance.

Before leaving school children should understand the principle underlying Compound Interest and find out the time necessary for a sum of money to double itself at given percentages. For form for calculations see Chapter X, p. 156, but apart from gaining a grasp of the idea, the work is of little value, as adults use tables and only the direct questions are soluble by other means.

Older children, especially boys, should gain some idea of the running of a company, that is, of what happens to our money when we invest it. A good plan is to dramatise this by having our class make-believe to run some concern inside school, *e.g.* a company to publish a school magazine, or to open a new light railway in the neighbourhood. A group of them can formally decide on the undertaking; invite "the public" of the class to subscribe the capital; discuss the spending of this capital on the stock necessary to begin work; analyse the running of the work for a year, distinguishing between income or gross receipts and expenditure, and in the case of expenditure between wages, upkeep of stock and other working expenses, a contribution to a reserve fund, and the balance. The children can next study the treatment of this balance, how a dividend of so much per cent. is "declared" from it on the capital subscribed by the public; each subscriber can calculate how much dividend he should receive. Later arises the question of some member of the public who needs his money again. He cannot have the capital, as it is now represented by things used in the concern instead of by money: children will suggest what he can do, and the idea of selling his stock, with the right to dividends, can be worked out. Selling prices above and below par will be discussed, with possible reasons for each and for measuring each by percentages. A description of a stock exchange will interest boys, in connection with buying and selling, but the whole thing should be treated on broad lines, while the intricacies of brokerage are beside the point. The main thing is not to be able to work tricky questions on stocks and shares, but to have a good general notion of how such concerns are run, of how prices rise and fall, and in the case of an intelligent class, probably of what is meant by "speculation." *The Pit*, a novel by Frank Norris, gives a very good description of the Chicago Wheat Exchange, which may prove suggestive to the teacher of this subject. When questions of the ordinary type are taken, it is wise occasionally to use a newspaper to bring up to date the facts they embody. Using "to-day's newspaper" gives an atmosphere of reality which is often sadly lacking in working "stocks and shares."

Citizenship

The main idea here is that with advancing civilisation and the concentration of individuals in communities small and great, many services which individuals once rendered to themselves are now rendered to them by the community as a whole. These services cost the community money which has to be repaid by the individuals on some plan or other.

In the primitive stages of life, for instance, each family finds its own water and light: a time comes when a small community share, say, a village pump or a harbour light by the sea: nowadays, communities of any size beyond the small village or hamlet provide a water supply and means of lighting for all the individuals concerned, until at the highest level we find a corporation like Manchester spending enormous sums in bringing a plentiful supply of pure water from the Lake District, and also providing electricity or gas for all parts of its mighty whole. Other services rendered to individuals by the town or county communities are police protection, universal elementary education and a limited amount of secondary education; care of the penniless, the insane, and those suffering from infectious diseases; making of roads and provision of public parks and sometimes of tramways. On the other hand, the great national community provides us, among other things, with postal and telephonic services, along with military and naval protection. But the bill has to be paid, often with excessive grumbling; when we pay our rates and taxes, we are apt to remember only the money we pay, not the services we gain!

What, then, are we to teach our pupils concerning these things? It matters most, perhaps, that they realise their privileges and responsibilities. If children felt that such things as streets, lamps, parks and even police belong to them and exist for their benefit, they might be a little more careful of their own property. It is, for instance, very important that children's original ideas of the policeman as a "bogey" to frighten naughty children should give way to the idea that he is there, not primarily to punish the evil-doer, but to protect those that do well. Girls in some districts sorely need this protection at times, and to give them a correct idea of the functions of the police matters not a little. Let our pupils count up the services they receive from the community, including even their education! We can start best, probably, either by an arresting question such as, "Who pays the policeman?" or by an historical survey, say, of the ways in which people have got their water supplies or provided for coast defences.

Next comes the cost of these services. On the analogy of the family, where the young and the infirm share in what is provided by the able-bodied adults, the State and the corporation or county do not demand payment from all. The plan of the latter is to estimate the total expenditure for a year and ask each *house* to pay a share. Children will see that for all houses to pay the same amount would not be fair, and so come to the idea that each house pays according to its rent. If the total paid in rents is known, say £100,000, and the total expenditure is known, say £10,000, the rate charged must be one-tenth of the rent; *i. e.* for £1 of rent the householder must pay 2*s.* of rates. It is easy, then, to calculate the total rate any one must pay.

Questions which the children should consider are (i) the relative value of the two plans of paying the rent alone to the landlord and the rate to the corporation, and of paying the landlord a sum to cover both rent and rate and leaving him to settle with the corporation: (ii) the proportion in which the rate paid is divided between poor rate, water rate, education rate, and other services, for this purpose using the rate collector's slips commonly issued: (iii) the distinction between the rental at which a house is "assessed" and the actual rent which may be paid.

In the case of the nation the money is collected partly by "taxing" various commodities used by the people, *e. g.* spirits, tea, sugar, tobacco, the amount of tax paid thus depending on the amount of the commodity used; partly by charging for some services (*e.g.* postal service and telephonic communication) as they are used, as corporations charge for householders' use of gas or electricity; partly by lump sums paid on incomes, or on amount of money bequeathed after death, or on increased values of property. The most important of these is the income tax: questions dealing with this should be taken in detail, keeping close to the actual facts of abatement for low incomes, for life insurance and annuities, and for possession of young children. We should base the work on the schedules sent out by the assessors.

Insurance

This subject has had little attention in Arithmetic lessons, partly because the great democratic developments of it are very recent and partly also because the Mathematical problems to which it gives rise are often very complicated. The whole theory of probabilities and much statistical work lie behind any scheme of insurance. Nevertheless, because insurance schemes touch all of us in these days, children should realise the main idea lying behind them all.

The main idea is the simple one that the accidents of life do not befall us all: we know not which of us they will befall: we should prepare ourselves in advance to meet them, but much less preparation will suffice if we join together and help each other than if we all act as separate individuals.

A good starting-point is to let the class make-believe to be a group of men and women earning their own living. Let each member of the class decide what his weekly wages are. Raise the question of any dangers to which their livelihood is exposed: they may suggest sickness or unemployment or accident. Simplify the situation by taking, say, sickness. Discuss who have been ill in the last year, for how long they were ill, and what would have been the cost of supporting them while they were sick. Then let them look ahead and discuss who will be ill within the next year; let them realise their ignorance of this and discuss how each means to meet sickness if it does come. They will advance to the idea of all paying a little to make a fund from which to pay money to sick who cannot support themselves. In the simple case in question they can work out from the previous year's statistics, allowing a sensible margin, the amount of insurance each person ought to pay per year or per week to have a right to a certain sum of money per week if ill. Compare the result with the conditions of the National Insurance Act with regard to both sickness benefits and premiums; this will emphasise the value of great combinations, of employers' compulsory contributions, and of State aid. Let them then suggest other accidents of life to meet which the insurance system is extended. Personal knowledge of the social conditions of his pupils will in this connection be most valuable to the teacher.

Bankruptcy

A very simple example will serve here. Mary owes 5*s*. to Helen, 3*s*. to Muriel and 2*s*. to Nan. A time comes when they all ask her for their money; the value of her possessions is found to amount to 5*s*. She pays Mary 5*s*., but gives nothing to Muriel or Nan. When children are asked to criticise this arrangement, they condemn it as unfair and quickly suggest each girl having only half of the money due her, since Mary possesses only half of the value of her debts. The terms "debtor," "creditor," "debts" or "liabilities," and "assets" can be introduced naturally, while the usual way of measuring Mary's ability to pay can be noted, that is, taking £1 of debt as the standard amount. In this case Mary can pay 10*s*. in £1 and so 2*s*. 6*d*. on 5*s*., 1*s*. 6*d*. on 3*s*., 1*s*. on 2*s*. When the terms are familiar and this plan has been accepted as the fairest possible under the disastrous circumstances, the usual type of question set presents no difficulty. This subject is, however, of very slight importance compared with the other topics treated in this chapter.

CHAPTER XV

GEOMETRY IN THE ELEMENTARY SCHOOL

In considering the place of Geometry in the Elementary School, we shall start by glancing at the scope of this subject and by outlining the history of its development in the race. This should help us to see the matter in truer perspective.

The derivation of "Geometry" is simple: "metry" = measurement, "geo" = earth (cp. Geography and Geology). Geometry is, in its original sense, earth measurement or, as it is sometimes called, mensuration. We note at once that most people have some of this measurement to do. Farmers and gardeners need it in connection with rents, seeds, manures, stock; surveyors and engineers need it in connection with town-planning, ship-building, water-works, road-making, railway and bridge-building, machinery of all kinds; architects, masons, plumbers, joiners, slaters and decorators, all need it for their own peculiar contribution to house-building; soldiers require it for scouting and artillery work, while it underlies all strategical manœuvres; sailors require it for steering their courses; designers and weavers of patterns on any kind of material start from Geometry, and the housewife uses it in cooking, sewing, decorating and gardening. In fact, almost the only work which does not demand some ability in earth-measurement is teaching! This is one reason why Geometry in schools is usually so much detached from the realities of life outside the school. The ground indicated above has to some extent already been covered in dealing with weights and measures; the question raised here is what additional ground should be covered in the Elementary School. Certainly Geometry in some form and to some extent is necessary as preparation for future life.

Historically, Geometry is a very ancient branch of

Mathematics, its earliest traces being found among the Egyptians and Phœnicians as long ago as 2000 B.C. We have a papyrus of that date, attributed to Ahmes, which includes rules for finding the cubical contents of barns and the areas of various figures, and also a method for the exact orientation of a temple which is the germ of the important Geometrical theorem about the square on the hypotenuse of a right-angled triangle. The story of Joseph having granaries built to store the grain of the seven full years in preparation for the seven years of famine, suggests a reason for the Egyptian interest in barns; the annual inundation of the Nile, which swept away landmarks and made a re-division of the adjoining land essential, suggests why the Egyptians were urged towards measurement of areas, while their religious development and their sun-worship explain their intense desire to find a line due East and West. To do this they found a North–South line from the positions of the heavenly bodies and then succeeded in marking a line perpendicular to this line by using a circle of rope knotted in three places so that it was divided in the ratio $3:4:5$. If sticks are fastened into the ground through the knots at each end of, say, the length measuring 3 units, on a line due North and South, a stick passed through the third knot and fastened into the ground so that all the string is taut, will put the line measuring 4 units in a direction perpendicular to the line measuring 3 units, since $5^2 = 4^2 + 3^2$. Thus the temple could be built to face due East. The results given in this old papyrus of Ahmes are frequently inaccurate; this is natural, as they were sought for practical purposes and discovered from a consideration of particular cases, not by general reasoning of any kind.

No great advance is found until the period 600 to 300 B.C., when the Greeks, with their acute minds, began to ask theoretical questions about this practical knowledge of Geometry. For instance, Ahmes knew that in one case the square on the hypotenuse of a right-angled triangle equals the sum of the squares on the other two sides; by 500 B.C. several other special cases were known, but Pythagoras, about this date, was the first to ask if this result would always be true, and if so, why. He did not set down any concise, synthetic proof, placed as an item in a well-knit system of geometrical theorems, nor did he attempt to prove a converse, but he explained by reasoning what had before been taken for granted and he generalised the knowledge previously limited to particular cases. Many of the well-known easy theorems about isosceles triangles, the sum of the angles of a triangle and similar triangles were explained in this somewhat unsystematic way by such men as Pythagoras and Thales. We read, for instance, that when Thales discovered how to inscribe a right-angled triangle in a circle, he sacrificed an ox to the immortal gods!

The third great advance follows close on this and centres round Alexandria, where what might be called the first university of the world was established about 300 B.C. Here Euclid was the "Professor" of Mathematics, a teacher so successful that, through all the centuries until within the last few years, the teaching of Geometry has been dominated by his work and, unfortunately, by a mistaken view of his methods which was accepted as the whole truth. Within recent times a book of his has been brought to light, written for his pupils. This contains ninety-eight illustrations of the application of analytical methods to the solution of problems by deduction, showing that Euclid tried to stir up his students to solve riders on the analytic plan adopted by any good teacher of to-day, and also that he considered the development of ability to prove new Geometrical truths for oneself the most important part of the teaching of Geometry. What, then, is the place of the well-known book entitled *Euclid*? Euclid's department flourished so much that in course of time he had many assistant teachers to whom he felt the necessity of giving some guidance in their teaching, especially guidance as to the body of Geometrical truth which should be known as a foundation for further work and also guidance as to its inter-relations. For men, then, to whom this knowledge was already living and growing, he systematised the groundwork of Geometrical knowledge in several books, setting the proofs down in a most concise, synthetic form, a form well suited to a summary of work already familiar. But Euclid's book for his pupils was lost; his book for his teachers lived. And in time, teachers forgot that it had been written as a manual for teachers and tried to impart it to all their pupils, just as it was. Only the few acute mathematical minds could flourish on a diet so indigestible; it is tragic to think of Euclid's feelings could he have walked round the schoolrooms of the last two thousand years! He himself taught Geometry to people whose interest in Geometry had been aroused through practical problems and through exercising their brains on the generalisation of results so obtained; he taught Geometry to them on analytic methods, working back from what they were required to prove to what they knew, in due course, after this had become familiar, having it set down in a usefully concise, synthetic form, but aiming mainly at the development of ability to apply the new knowledge and the new analytic method to the solution of problems still unsolved. Men ever since, believing they followed in the steps of a great master, have taught a mere skeleton of Geometrical knowledge, deprived of all the living associations which give it vital interest for the ordinary individual; they have most frequently taught a brief summary of the ground covered in a form so much curtailed that the average student took refuge in learning it by heart; in doing so, they failed entirely to develop, in any save the really mathematical type of mind, ability to apply old knowledge to the solution of new Geometrical problems. Within the last twenty years we have wakened to the tragedy of this and are still battling in the whirlpool of change. More knowledge of the historical development and of the mistake underlying our methods of the last twenty centuries should hasten the return of our teaching of Geometry to a more stable equilibrium. Those wishing further knowledge than this bald outline supplies are referred to Ball's *History of Mathematics*, or Cajori's *History of Elementary Mathematics*, or Benchara Branford's *Study of Mathematical Education.*

As regards the further history of Geometry, no great change took place until the modern developments of the nineteenth century, developments so far beyond the pupils of the Elementary School that we entirely omit them and hasten to draw from the above outline the conclusions we require.

Corresponding to the three stages of the historical development of Geometry, three aspects of the teaching of Geometry arise for consideration—

(*a*) As Geometry grew, in very early times, side by side with the beginnings of Arithmetic, the recent development of a little Geometrical work even in the lower standards of Elementary Schools has the justification of history behind it. But history takes us farther by supplying tests for our choice of material and for our methods. The Geometry taught should grow out of problems of living interest to the pupils, more particularly out of problems which arise in connection with handwork of various kinds; it should be taught in such a way that the new knowledge imparts new skill. The matter taught will differ from school to school in accordance with the nature of the handwork scheme, but it is justified only if it is related to life and leads to power.

(*b*) As children's minds grow more active and inquisitive, they should not be restricted to particular cases of practical problems. They should be encouraged to ask questions about results found true in one case: will these results be true in other cases? They should try to explain their reasons for their views in their own words, their methods of explanation being criticised and gradually improved. They should also attempt to solve other problems by the use of such knowledge as they have obtained from their practical work and its discussion.

(*c*) When our pupils' minds have reached the stage of development when abstract reasoning is possible and is instructive, they should be led to criticise more thoroughly the forms and soundness of their explanations of Geometrical truth and so systematise and expand the old knowledge. This means finally the building up of a system of Geometrical truth, by rigorous proof of a deductive kind, accompanied by an appreciation of the value of such a logical treatment of Geometry.

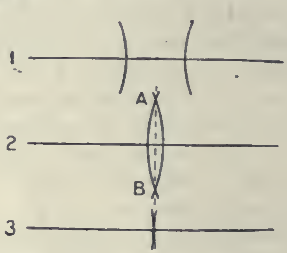
Obviously, as long as the compulsory school age does not rise above fourteen, only (*a*) and (*b*) need be considered in the Elementary School. We shall give a few suggestions as to each, purposely omitting any detailed working out of a scheme, since each school should make its own, based on the scope of its general curriculum.

As regards a practical line of approach to Geometry, introductory notions of simple Geometrical figures, or what might be termed simple descriptive Geometry, can be developed in connection with paper-folding, or the use of some of the Montessori apparatus, or the making of objects in paper, cardboard or wood. The making of objects in various materials, practical Geographical problems in connection with finding the altitude of the sun or unknown directions or inaccessible heights and distances, the making of plans and maps, the keeping of a school garden, all these give rise to Geometrical problems which, when solved, give the child greater power in dealing with the original subject. It may be simplest here to centre our suggestions round the use of the simple Geometrical instruments.

Work with the ruler has already been taken in some detail in Chapters VII. and X. To this may be added the making of plans for models; this involves measuring the actual object to be copied, representing it by drawing, and planning the model on the correct scale before cutting it out and actually constructing it. Any of the modern books on cardboard or wood modelling give models which can be used in this way. We should also include the drawing of plans of the classroom, the school and the neighbouring district, with the resulting measurement by a scale. In such work children quickly learn the value of accuracy in measurement; a lid that refuses to fit or a plan that gives wrong information inculcates accuracy much more forcibly than daily sermons on the subject!

Such work leads naturally to the use of the set-square and the common facts connected with right angles. The class, say, has made a match box and found that its sides are not vertical, or that a plan gives wrong information, their mistakes being due to their angles not being "square." The idea of an angle as measuring revolution, or amount of turning, along with the ideas of a complete revolution (illustrated, say, by hands of a clock or by children's movements), half a complete revolution or a straight angle, quarter of a complete revolution or a right angle, can be introduced; children can then make simple set-squares for themselves by taking any scrap of paper, folding it once and then a second time, edge on edge, to secure a quarter of a complete revolution. To substitute a more substantial set-square of cardboard or wood is a simple advance; the improved results in box-making and plan-drawing will be sufficient evidence of its value. There is no reason why children should not make the acquaintance of halves and thirds of a right angle and use these in their constructive work, by means of the other corners of their set squares. Easy problems on the movements of clock hands, of the sun and of children themselves, will familiarise our pupils with the new ideas. Triangles of the common types represented by the set-square shapes can be studied.

A little later, children can be presented, say, with a hexagonal box or tray and allowed to construct this as they like. They find it hard to do this accurately and so respond well to a suggestion to leave the work for a time and see if new knowledge about hexagons can be discovered. The connection with a circle immediately suggests itself and introduces compasses. The gardener's plan of making a circular bed by means of two sticks and a string leads naturally to the more elaborate pair of compasses, with the fixed point as "centre," the fixed distance between point and pencil as "radius," and the pencil swinging round to make the "circumference." An abundance of interesting work now presents itself. Children can study hexagons and six-pointed stars, find out that the distances between consecutive points are all equal and, moreover, equal to the distance of each from the centre of the figure or to half the distance across the figure. From this they can invent a way of making a hexagon and a star by the use of compasses, advancing to the invention of Geometrical designs. Children do surprisingly good colour work in this direction and their designs can be applied as decoration to needlework or stencilling or other forms of handwork. They can also be asked to find out, as a puzzle, how some of the familiar linoleum and wall-paper designs were constructed. Geometrical knowledge about equilateral triangles, about hexagons, about the making of an angle of 60° (the "degree" as a small unit equal to $\frac{1}{90}$ of a right angle becoming useful at this stage), and about the relationship of the circumference of a circle to its diameter, here arises naturally. Children can apply their new skill to making first-rate hexagonal boxes and clock faces for use either by themselves or by a lower standard. The clock face leads to the necessity for halving the lines joining the six hours first found by the hexagonal construction. If children try to guess the middle point by use of compasses, they make diagrams like—



Instead of continuing to mark points until they finally reach diagram (iii), they will see that (ii) can be used immediately if the points A and B be joined; such appears to be the way in which the familiar method of bisecting a straight line arose. When a construction for bisecting an angle is required a little later, it can easily be invented by a slight modification of the above.

Various lines of approach to the protractor may be taken. The making of a pentagonal box or tray introduces angles not familiar as simple parts of a right angle and so emphasises necessity for new knowledge. An attempt to measure some inaccessible distance, say, the distance of a ship from the shore; a study of the altitude of the sun; the exact description of some direction for naval or military purposes—all these lead to the necessity for further skill in measuring angles less than a right angle. In whatever way the need for a protractor comes to be felt by the child, he should always begin by constructing a fair-sized cardboard protractor for himself; if this be done, half the difficulties met with in teaching the use of the protractor disappear as if by magic! Children can begin with a semicircle, and make in it the angles they know how to make—60°, 90°, 120°. They can get 30° and 150° either by using the arc between 60° and 90°, or by developing a method of bisecting an angle of 60°; they find the arcs for 15° by bisecting an angle of 30°; the smaller arcs necessary for 5° can be guessed by the eye in such roughly made protractors. The angles themselves should be drawn, at least in part, instead of merely marking the arcs, as otherwise children do not understand the fundamental principle of the protractor. They should finally complete their protractors by marking the centres clearly, by cutting away the unnecessary heavy cardboard so as to leave either the usual semicircular or rectangular protractor, by marking every 10° strongly and every intermediate 5° less strongly and by making scales to read from both left and right on any one of the plans commonly adopted.

The need for acquiring skill in the use of a protractor gives the child an opportunity of discovering, by measurement of particular cases, many of the well-known truths about angles, triangles, parallelograms and circles. Here, if not before, stage (*b*) in Geometrical teaching can be begun, by asking children to discuss, and then finally explain *in their own words*, the reasons, say, for the equality of vertically opposite angles, or for the sum of the angles of a triangle being always two right angles, or for the equality of angles in the same segment. This work will resemble not at all the "proofs" usually taken with older children; it is simply an introduction to such work. Easy problems, more especially easy numerical problems, involving this new knowledge, should be set frequently. But when skill in using a protractor has been acquired, it should be applied to a variety of objects. Amongst others, we can take the mariner's compass; an extension of plan-drawing, of Geometrical designing and of making of trays or boxes to include regular polygons not yet dealt with; the use of the plane table and of triangulation to make a map of some piece of ground; the measuring of inaccessible heights and distances and of gradients by the use of protractors specially devised to measure angles from a vertical or horizontal line and to provide a simple means of "sighting"; or the finding of one's direction at any moment by the use of a watch and the sun.

Some of the work outlined above should introduce our pupils to the use of the simple Geometrical instruments, while satisfying the principles of Geometrical teaching enunciated earlier in this chapter. Other Geometrical work of value in Elementary Schools includes the introduction of the results of the theorem of Pythagoras about right-angled triangles, the study of similar triangles and polygons, the making of models of the simpler mathematical solids, the measurement of areas and volumes and the development of the rule for extracting a square root. The next two chapters will be devoted to a more detailed treatment of area, volume and square root.

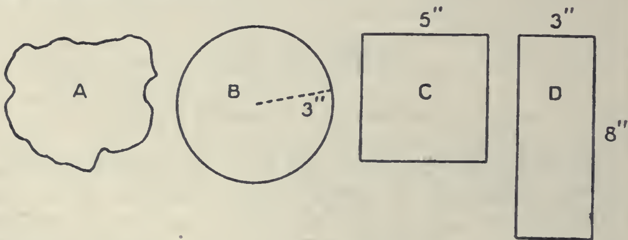
CHAPTER XVI

AREAS AND VOLUMES

THE real difficulty found by many teachers in securing satisfactory results on these topics seems to arise from confusion in our pupils' minds as to the distinctions between lines, surfaces and solids, and between length, area and volume; such confusion leads at once to inaccurate numerical results owing to use of the wrong table. Another cause of trouble is that many of our pupils apply the rule for rectangular areas (area = length × breadth) in a way both mechanical and reckless, heeding not its limitation to rectangular forms. To secure good results, it is absolutely necessary at first to make haste slowly and to develop the fundamental ideas with great thoroughness. Fortunately the measurement of volume seldom presents much difficulty if the measurement of area has been placed on a sound basis.

Measurement of area is now frequently introduced even in Standard II. Children at this stage are well able to describe surface as the "top" of anything and to indicate various lines (straight or curved) and surfaces (flat or rounded or uneven) in the room and on themselves. They can also learn that surfaces must be measured by a unit which is a surface and that a square inch is a very useful unit for measuring small surfaces; they can measure any small surface approximately by covering it with square inches and then, by applying this method to the rectangle, they can establish the rule for a rectangular area and apply it to the measurement of the areas of various rectangular surfaces. Since similar work—taken, of course, more rapidly—is the best way of introducing or revising the beginnings of areas in later standards, it may be worth while to indicate suitable steps.

(i) Present class with pieces of paper of various shapes and sizes and ask a question which will lead to measurement of their surfaces.



e. g. These represent four pieces of cake, all of equal thickness; which would you prefer to have?

Or, These represent four fields of equally good soil which you may rent for the same sum of money; which would you choose?

(ii) From the medley of guesswork answers received, it is possible to develop the distinction between a surface and a line, between, *e. g.*, the fence round the field and the soil inside it, so that the problem, "How to measure a surface" definitely emerges.

(iii) From the medley of suggestions as to ways of measuring the given surface, it is simple to lead up to the idea that a measuring *surface* is necessary, that a *square* is a good shape for it, and a square *inch* a convenient size.

N.B.—Refuse to accept length × breadth, which some child is almost certain to have heard before, unless such a reason for it is given as will satisfy the class. It does not apply to A and B.

(iv) Let class make paper square inches, distinguish between an inch of *edge* and a square inch of *surface,* and then proceed to answer question in (i) by fitting on the squares and counting them. The class can be divided into four groups for this purpose.

(v) The subsequent steps must vary greatly from class to class. Some teachers will repeat similar questions, being doubtful if the idea of surface is sufficiently clear. Some will present a similar question involving larger surfaces and so introduce at once square feet and square yards. Some will fasten at once on the fact that in the question set, C and D were most easily dealt with, and so invite class to study the areas of rectangles, pointing out that this method of cutting out squares and fitting them on is slow. When the rectangle has been reached we advance to (vi).

(vi) By drawing rectangles, children can see that they can count the number of square inches in their surface by drawing lines to divide them into square inches. This is simpler than fitting on squares.

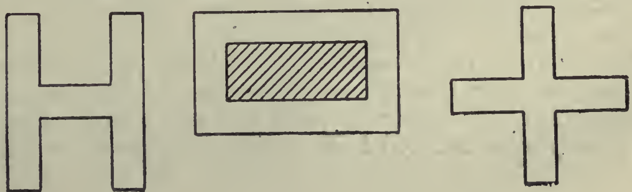
(vii) Soon they can tell the answer without counting. The length gives the number of square inches in one row; the breadth the number of rows. Therefore the number of square inches in the area is obtained by multiplying the number of inches in the length by the number of inches in the breadth. Later this can be shortened to—

$$\text{Area} = \text{length} \times \text{breadth};$$

still later to

$$A = lb.$$

When this has been achieved, the subsequent work is straightforward. It must include, in whatever order seems best to the teacher, the building up of the relationships between square inches, square feet and square yards; the application of the rule for a rectangular area to finding the areas of real objects, *e. g.* room, window, playground, handkerchief, book (for a suggestion as to a useful type of lesson see p. 98); the application of this rule to the working of written problems of various kinds; the extension of the rule to cover fractional quantities by drawing of diagrams (*e. g.* $5\frac{1}{2}$ ft. $\times$ $3\frac{1}{2}$ ft.) and its application to further written problems, these problems being connected as closely as possible with girls' work in housecraft; the application of the rule to finding the areas of surfaces easily subdivided into rectangles, such as—



In dealing with housecraft problems, interesting and useful work on suitable prices, materials and even colours can be introduced; from the arithmetical standpoint, the main requirement is that the methods used be such as to give a sensible result in house management. For instance, in order to discover how many tiles to order for a hearth, it is not safe to find the area of the hearth and divide it by the area of a tile. If the resulting number of tiles was ordered, it would be possible to cover the hearth with tiles, but, in nearly every case, only by breaking a few tiles into fragments! The wise plan is to count the number of tiles required for each row, and the number of rows. Similarly, in trying to discover how much matting, or linoleum, or other material laid down in strips, is required for a floor-covering, to divide the area of the floor by the width of the material and purchase the resulting length, is apt to make it necessary to cut one's last strip into many small pieces in a fashion that is sometimes too unsightly to be possible. To get an absolutely right result, we must find the number of strips required (by dividing width of room by width of material) and the length of each strip; and use the narrow piece left over for some other purpose. The economical housewife can sometimes compromise between the two, arranging to have a few simple joinings in a corner of the room where they are not too conspicuous. The essential point is that the attitude of the girls to such problems be that of common sense and wise economy. Practical lessons are essential.

Finding the correct amount of wall-paper is another operation requiring judgment and a sense of the fitness of things. To find the total wall area is usually simple: multiply the perimeter of the room by its height. To find the area of a "piece" or "roll" of paper is also easy; obviously the division of the one area by the other will give an approximate answer to the question, "How much paper must I buy?" since paper can be bought only by the "piece" or "roll," usually 12 yds. long. Only common sense can turn this approximate answer into a sensible answer! If the paper be plain or have a tiny pattern, probably what is saved from windows and doors, with the possible extra fraction of a piece (*e.g.* from buying 8 pieces, if above answer is $7\frac{3}{7}$ pieces) will make sightly joinings possible; if there are many doors and windows, it may even be wasteful to buy so much. But with very little door or window space, or with a large pattern awkward for joinings, one or two additional rolls must be purchased. This work can often be made most interesting by obtaining old books of paper patterns from decorators.

In the above work we have already hinted at ways of passing beyond the measurement of rectangular areas. We now summarise very briefly the ground still to be covered to gain complete skill in measurement of area.

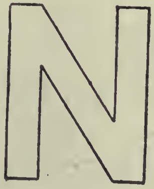
(a) *Parallelograms.*

(i) By cutting out, change the parallelogram to an equal rectangle.

$\therefore$ Area = base $\times$ height.

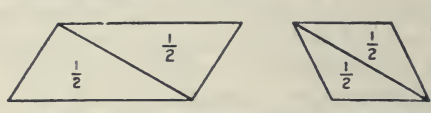
(ii) Test the rule by drawing parallelograms on squared paper and finding their area by counting squares. In counting, squares less than half a square are omitted; those greater than half a square count as one.

(iii) Apply to numerical examples, including cases of complex figures: *e.g.*



(b) *Triangles.*

(i) By paper-cutting and folding, show that every triangle is half a parallelogram; this includes obtuse-angled triangles.

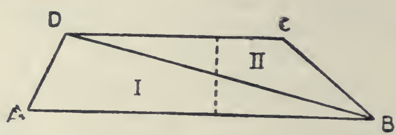


$\therefore$ Area of triangle $= \frac{1}{2}$ base $\times$ height.

(ii) Test result by use of squared paper.

(iii) Apply as above: *e. g.* gable end of a house, kite.

(c) *Trapezium.*

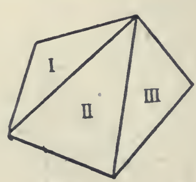


Area = triangle I + triangle II.
$= \frac{1}{2}$ AB $\times$ height of trap. + $\frac{1}{2}$ CD $\times$ height of trap.
$= \frac{1}{2}$ (AB + CD) $\times$ height.
$= \frac{1}{2}$ the sum of the parallel sides $\times$ distance between them.

Apply this to roofs of houses, bay windows and, most of all, to (*d*) (iii) below.

(d) *Any Polygon.*

(i) Divide into triangles and add their areas.



(ii) Transfer to squared paper and count the squares.

(iii) Use a base line and offsets, dividing the polygon into right-angled triangles and trapezia. This is the surveyor's method—

	AB = 200 yds.	
	AK = 40 „	KC = 80 yds.
e. g. LF = 20 yds.	AL = 50 „	
	AM = 140 „	MD = 60 yds.
NE = 40 yds.	AN = 160 „	

$$\text{Area of ABCDEF} = \text{I} + \text{II} + \text{III} + \text{IV} + \text{V} + \text{VI}$$
$$= \left(\tfrac{1}{2} . 40 . 80 + 100 . \tfrac{80+60}{2} + \tfrac{1}{2} . 60 . 40 + \tfrac{1}{2} . 40 . 40 + 110 . \tfrac{40+20}{2} + \tfrac{1}{2}\, 20 . 50\right) \text{ sq. yds.}$$

(e) *Any Irregular Surface.*

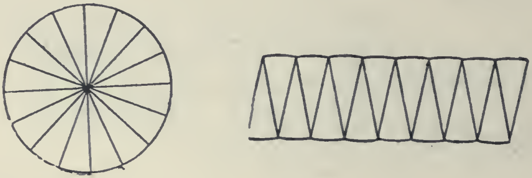
(i) Reduce to approximately equal polygon and apply methods of (*d*) above.

(ii) Cut out of paper; cut out a known area from the same paper. Weigh both. The areas are proportional to the weights; hence find area of unknown surface.

(iii) Transfer to squared paper and count squares.

(f) *Circle.*

(i) Change circle into a figure like a parallelogram by cutting it into sectors and arranging them thus—



The greater the number of sectors, the more nearly do they make a parallelogram.

$$\begin{aligned}
\therefore \text{ Area of circle } &= \text{area of parallelogram} \\
&= \text{base of } /\!/^{\text{m}} \times \text{height of } /\!/^{\text{m}} \\
&= \tfrac{1}{2} \text{ circumference of } \odot \times \text{radius of } \odot \\
&= \tfrac{1}{2} \times 2\pi r \times r \\
&= \pi r^2
\end{aligned}$$

(ii) Test by drawing on squared paper and counting squares.

(iii) Apply to numerical problems, including areas of rings, flat ends of cylinders and cones.

(g) *Curved Surface of Cylinder.*

(i) Cut out paper to cover this surface exactly. Paper is a rectangle, breadth equal to height of cylinder, length equal to circumference of cylinder.

$$\begin{aligned}
\therefore \text{ Area of curved surface } &= \text{area of rectangle.} \\
&= \text{circumf. of cyl.} \times \text{ht. of cyl.} \\
&= 2\pi r \times h \\
&= 2\pi r h.
\end{aligned}$$

(ii) Apply to pipes, chimneys, legs of chairs and other familiar objects.

The introduction to measurement of volume can be taken on lines similar to the introduction to measurement of area, by setting a problem involving the discovery of the cubical contents of differently shaped vessels and boxes. Cubic inches are not so easily provided as square inches, but frequently inch cubes can be borrowed from the infant department of the school; or our pupils can make paper cubes for themselves.

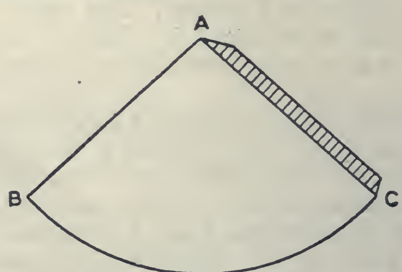
This work leads naturally to the discovery of the rule for measuring the volume of a rectangular box by taking the product of its length, its breadth and its height; or if two of the dimensions are replaced by the area of the surface they define, by the product of this area and the third dimension. These rules can be applied to the practical measurement of cubic contents and also to the solution of written problems. One useful application is to study simple questions on ventilation, depending on a knowledge of the air supply necessary for each individual. The children can compare the air-space per individual in different classrooms. Cubic feet and cubic yards will be required as well as cubic inches; the table is easily built up from the table of length.

The measurement of the volume of any solid object insoluble in water, can be found by immersing it, when it displaces its own volume of water. The rise in water can be measured either by use of a measuring-jar or by use of the pipette and burette. The cubic contents of any vessel may be measured by filling it with water or sand and then measuring the water or sand by a measuring-jar.

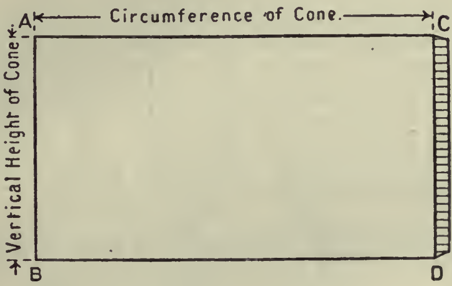
But in certain cases volumes can be calculated by rules almost as simple as the rules for a rectangular prism. Before closing, we shall indicate ways of establishing these rules through practical work. In no case does the practical work of itself *prove* the rule; but it gives our pupils good reason for believing our assertion that mathematicians have proved the rule true. We shall first consider the cylinder and the cone.

The cylinder, of itself, is simple. If it is 5 in. high, say, we can imagine 5 layers of cubic inches filling it. How many cubes will there be in each layer? Obviously, one cube can rest on each square inch of its base, or, in other words, the number of cubic inches in one layer is the number of square inches in the base, which we already know to be πr^2. Thus the volume in this case is $5\pi r^2$, in the general case $\pi r^2 \times$ ht. or $\pi r^2 h$. Children should apply this formula to finding the volume of common cylindrical objects—tins, for instance, of various kinds. They can find out how much cheaper it is to buy cocoa in a large tin rather than in a small. They can also illustrate the truth of the formula by taking a muffin or some other cylindrical object easily cut, dividing it into wedge-shaped sectors and arranging the sectors as the sectors of the circle were arranged on p. 220. The more pieces there are, the more nearly will the solid made be a rectangular block; its width is the radius of the cylinder, its height the height of the cylinder, its length $\frac{1}{2}$ the circumference of the cylinder. Thus its volume $= r \times h \times \pi r = \pi r^2 h$. The correctness of results obtained by the formula can, in any particular case, be checked by immersing the cylinder in water as suggested above.

The volume of the cone is best measured by comparing it with its corresponding cylinder; that is, with a cylinder of the same radius and the same vertical height. To carry out this work, our pupils must first make a cone; the diagram suggests a simple method—

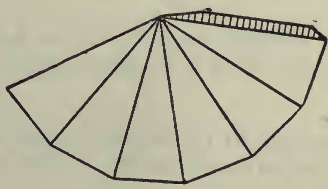


This sector is rolled up so that AC lies along AB, while the adjoining flap is gummed underneath. They must next make the corresponding cylinder. For this they must measure the circumference of the cone and the vertical height of the cone. The plan is as follows—

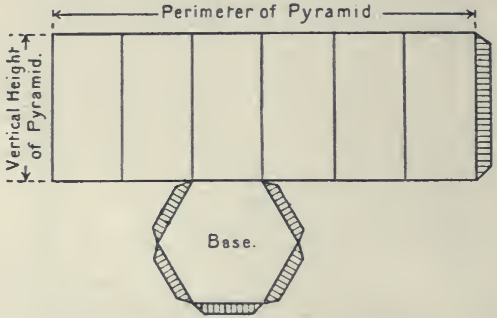


The rectangle is rolled up so that CD lies along AB, while the flap is again gummed underneath. A circular bottom is then attached. When the children have completed these two models, they can estimate for themselves the relation between their cubic contents and then check these estimates by filling them with sand or sawdust; they will find that, as nearly as possible, the cone must be emptied three times into the cylinder to fill it. Thus the formula for the volume of a cone is $V = \frac{1}{3}\pi r^2 h$, since r and h for the cone are the same as r and h for the cylinder. The formula should be applied to practical problems and the results obtained should be checked, as suggested above for the cylinder.

Exactly similar methods and reasoning can be applied to any pair of solids, consisting of a prism and its corresponding pyramid. The volume of the prism is the product of its base-area and its height; see reasoning for cylinder. Take case of hexagonal bases. A six-sided pyramid is made as suggested in diagram—

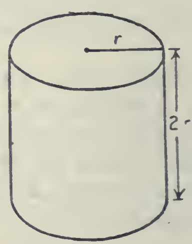


The side and vertical height of this is measured, so that its corresponding prism can be made, as indicated below—



If sand is now used, it will be found that the pyramid is one third of the corresponding prism; *i. e.* its volume is $\frac{1}{3}Ah$, where A is the area of the base and h the vertical height.

In conclusion, we suggest a rough method of justifying the usual formula for the volume of a sphere, by comparing it with the volume of a cylinder of the same diameter and of height equal to its diameter.



Volume of this cylinder is $\pi r^2 \times 2r$ or $2\pi r^3$. We try to show that the sphere is $\frac{2}{3}$ of this cylinder, and so $\frac{2}{3}$ of $2\pi r^3$ or $\frac{4}{3}\pi r^3$.

(i) Make in clay or plasticine a cylinder whose height is equal to its diameter. Call its volume C.

(ii) Cut away from top and bottom of cylinder, so that a sphere is left. Call the volume of the part cut away P and the volume of the sphere S.

(iii) Let class weigh the part cut away and weigh the sphere, in a number of cases. From the results obtained, within limits due to the roughness of the experiments, S is found to be about 2P. Tell class that this has been proved by mathematicians to be true, although they themselves have not proved it.

(iv) If S is twice P, S must be $\frac{2}{3}$ of the cylinder C.

(v) $\therefore$ volume of sphere $= S = \frac{2}{3}$ of C

$$= \tfrac{2}{3} \text{ of } 2\pi r^3$$
$$= \tfrac{4}{3}\pi r^3$$

(vi) Apply this formula to practical measurement and test the results by use of measuring-jar.

CHAPTER XVII

SQUARE ROOTS

The teaching of square root need not long detain us. The finding of a square root by factorisation is a simple matter; the Geometrical applications to finding the side of a square, given its area, and hence to finding the length of the hypotenuse of a right-angled triangle, given the other two sides, are familiar and useful. In some schools, however, it may be desirable to teach the older pupils the ordinary rule for extracting a square root. This rule: "To obtain the next figure, always divide by twice what you have already found," is simple to work, but teachers often find it hard to explain and still harder to teach in such a way that our pupils make the journey for themselves. We are indebted to Dr. Nunn for an introduction to a Geometrical way of approach which solves these difficulties. As many people are unacquainted with this, we do not apologise for here repeating Dr. Nunn's ideas on the subject. It will be simplest, perhaps, to divide the ground to be covered into steps suitable for teaching.

(*a*) Set questions of this type: ABCD is a square piece of ground; its area is 121 sq. yds. Find the length of AB.

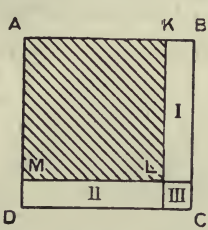
(*b*) Set questions of this type: ABCD is a square piece of ground; its area is 68·89 sq. yds. Find the length of AB.

(i) It is easy for our pupils to tell that AB lies between 8 yds. and 9 yds. Cut off AK = 8 yds.; note that length of KB must be less than 1 yd.

The main part of ABCD is thus AKLM

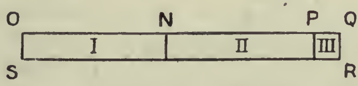
= 64 sq. yds. . . . . . . A.

(ii) What is length of KB? To find this, study portion left unshaded. We know about this part that KL = LM = 8 yds.; that KB = MD, each less



than 1 yd.; that the area of the part is (68·89—64) sq. yds., *i. e.* 4·89 sq. yds. . . . . B.

Let children change this remainder into a rectangle, since a rectangle is the simplest figure.



Compare the rectangle with the remainder of the square, to discover all that is known about the rectangle and what is wanted from it.

Area = 4·89 sq. yds.
Length = ON + NP + PQ = 16 + x yds.
where x = KB = less than 1 yd.
= 16 yds. approximately.

We wish to find KB, which is QR or PQ or x yards. To find the breadth of the rectangle, we divide its area by its approximate length.

Breadth will be about $\frac{4 \cdot 89}{16}$ yds., which is about ·3 yd. . . . . C.

If this were the length wanted, OQ = 16·3 yds., QR = ·3 yd. and the area would be 16·3 × ·3 sq. yds. or 4·89 sq. yds., which is correct. . . . . D.

Thus we have found the exact breadth, by good fortune!

∴ KB = QR = PQ = ·3 yd.

(iii) Therefore the length we set out to find, or AB, which is AK + KB, = 8·3 yds. . . . . E.

(iv) The five numerical calculations, in the lines marked A, B, C, D, E, are summarised below—

```
      8)68·89(8
        64
       ─────
16·3)4·89(·3
     4·89
     ────
```

$\sqrt{68\cdot89} = 8\cdot3.$

Geometrical key to above—

```
             AK)ABCD(AM
                AKLM
                ─────
     OP + PQ  )OQRS(PQ or QR
or 2×AK + PQ   OQRS
or OQ          ─────
          AB = AM + QR.
```

A. Number which gives nearest square below 68 is 8.

B. 68·89 − 64 = 4·89.

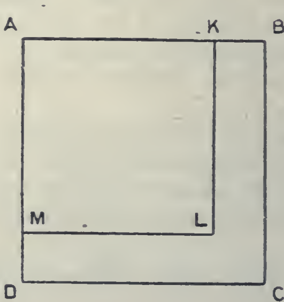
C. Divide 4·89 by twice 8, the number already found; the result appears to be ·3.

D. Divide 4·89 not by 16 but by 16·3; 16·3 goes into 4·89 exactly ·3, without remainder. Therefore search is ended.

E. Square root is 8·3.

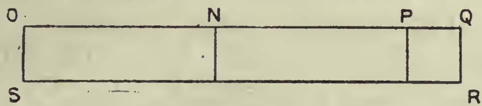
(*c*) Set questions of this type: ABCD is a square piece of ground; its area is 10·0489 sq. metres Find the length of AB.

(i) As above, AB lies between 3 and 4 metres, and main part of square is AKLM of area 9 sq. metres. KB is less than 1 metre . . . . A.



(ii) As above, area of remainder is 10·0489 — 9 sq. metres or 1·0489 sq. metres; KL = LM = 3 metres. Let KB = x metres . . . . B.

Change remainder into rectangle OQRS.



Area of rectangle = 1·0489 sq. metres.

Length ,, = 3 + 3 + x metres = 6 metres approximately.

∴ as above, QR or breadth is approximately $\frac{1\cdot0489}{6}$ . . . . C.

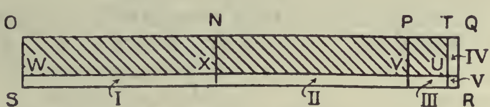
or some length between ·1 and ·2 metre. Try ·1.

Length of rectangle would be (3 + 3 + ·1) metre = 6·1 metres.

Breadth of rectangle would be ·1 metre.

∴ area would be 6·1 × ·1 sq. metre = ·61 sq. metre . . . . D.

But this is less than area of OQRS. It follows that PQ or QR is greater than ·1 metre but less than ·2, and our problem is not yet solved. We know only that the main part of KB is ·1 metre, but that KB is a little longer. Mark off part known on rectangle OQRS.



OT = 6·1 metres, TU = ·1 metre.

Therefore OTUW, the shaded area, is ·61 sq. metre.

(iii) What is the length of TQ? To find this study the portion left unshaded. We know that WX = XV = 3; that VU = UT = ·1; that TQ = WS = y, say, less than ·1 metre; that the area left is (1·0489 — ·61) sq. metres or ·4389 sq. metre . . . . . . E.

Let children change this remainder also into a rectangle and compare the new rectangle with the remainder of the first rectangle.

Area of this rectangle = ·4389 sq. metre.
Length „ „ = EF + FG + GH + HI + IY.
= (3 + 3 + ·1 + ·1 + y) metres.
= 6·2 metres, approximately.

We wish to find IY or YZ, which was TQ in previous figure.

YZ is breadth; breadth is about $\frac{·4389}{6·2}$ or about ·07 . . . . F.

If this were the length wanted,

EY = (6·2 + ·07) metres = 6·27 metres
YZ = ·07 metre.

and the area would be (6·27 × ·07) sq. metres or ·4389 sq. metre . . . . G.

But this is correct; and we have now found that YZ or TQ is ·07 metre exactly.

(iv) Therefore PQ or KB is ·17 metre, and AB is 3·17 metres . . . . H.

(v) The eight numerical calculations, in the lines marked A, B . . . H, are summarised below.

```
3)10·0489(3
  9
  ─────────
6·1)1·0489(·1
      ·61
      ─────
6·27)·4389(·07
      ·4389
      ─────
```

A, B, C, as before.

D. Divide 1·0489 not by 6, but by 6·1; 1·0489 ÷ 6·1 gives ·1, but there is a remainder. Thus search is unfinished.

E. 1·0489 — ·61 = ·4389.

F. Divide ·4389 by twice 3·1, already found; the result appears to be ·07.

G. Divide ·4389 not by 6·2 but by 6·27; ·4389 ÷ 6·27 gives ·07 without remainder. Search is therefore ended.

H. Square root is 3·17.

(*d*) If the above types are taken until the children can explain the written working without actually drawing diagrams, and an illustration is taken which does not work out even with third figure (so that there is a third remainder to be turned into a third rectangle whose length is approximately twice the lengths already found), the formal development of the rule need not be delayed. The only remaining difficulty may be with the crossing off of figures in pairs from the decimal point. By trial we can construct this table for numbers and their squares.

N (or $\sqrt{B}$)	N^2 (or B)
1 figure	1 or 2 figures
2 figures	3 ,, 4 ,,
3 ,,	5 ,, 6 ,,
....	....
n ,,	$2n - 1$ or $2n$ figures
1 dec. place	2 dec. places
2 dec. places	4 ,, ,,
3 ,, ,,	6 ,, ,,
n ,, ,,	$2n$,, ,,

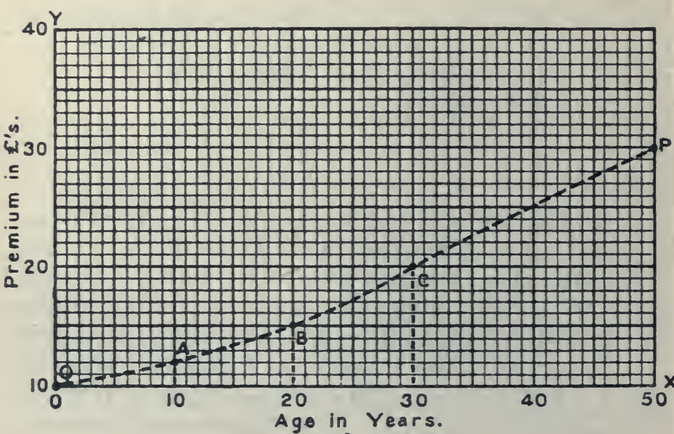
Now let N^2 be B and N must be $\sqrt{B}$. We see then that where B contains 1 or 2 figures, the square root of B contains 1 figure; where B contains 3 or 4 figures, $\sqrt{B}$ contains 2 figures, and so on. Hence figures must be counted off in pairs from the decimal point to the left. Also where B has 2 decimal places, $\sqrt{B}$ has 1 decimal place; B 4 decimal places, $\sqrt{B}$ 2 decimal places and so on. Hence figures must be counted off in pairs from the decimal point to the right.

This completes the rule as commonly known. It is wise to remember, however, that square roots are usually extracted by adults by the use of a slide rule or of logarithmic tables.

CHAPTER XVIII

GRAPHICAL WORK IN ELEMENTARY SCHOOLS

The value of the graphical method of representing simultaneous changes in two connected quantities has, within recent times, become so generally recognised that some acquaintance with this kind of work forms an essential part of even an Elementary curriculum in Mathematics. Before estimating its importance and indicating lines of approach, it may be helpful to take a glance at the nature of the subject under consideration.



We all understand the bare framework of the graph. Two quantities which change simultaneously (for instance, date and temperature, diameter and circumference of a circle, age of man beginning to pay life insurance premiums and the amount of the annual premium) are represented by distances measured along two lines or "axes," at right angles to each other, meeting at a point commonly called the origin.

Take the case of insurance premiums, where a man may start payment, say, at any age between 0 years and 50 years. His age would be measured along OX, starting from O. Our scale must run from 0 to 50. See diagram above. Suppose that the premium to secure a certain sum at death varies from £10 to £30. This premium would be measured along OY, starting from O, and the scale must run from £10 to £30. Since to age 0 years corresponds £10, 0 is one point on the graph. If to age 10 corresponds £12, A is a second point; if to age 20 corresponds £15, B is a third point; if to age 30, £20, C is a fourth point, and so on. Thus in any graph the essentials are—

(i) It must represent two quantities which change simultaneously.

(ii) These two quantities are represented by distances along two rectangular axes; therefore each axis must be labelled with the quantity it represents.

(iii) Since the distance along an axis is proportional to the magnitude of the quantity, a scale must be attached to each axis showing the value attached to the origin 0, and sufficient to indicate the values of all points on the axis in question.

(iv) In the graph, the values which the two quantities have at the same time are measured off along each axis and so fix a point: *e.g.* in above diagram where to age 30 years corresponds £20, we measure "30 years" along the horizontal axis, and from there measure "£20" in the direction of the vertical axis, arriving at the point C. C thus represents the fact that £20 and 30 years occur together in the set of circumstances under consideration.

But when we look at the whole matter more closely we find that graphs fall into three main classes, either corresponding to the ways in which the single points found as in (iv) above are grouped together, or corresponding to the types of connection between the two quantities which change simultaneously. Let us take the latter point of view.

(*a*) Some quantities are connected not because of any "kinship" between them, but simply because some one desires to think of them together. They are acquaintances, as it were, not relations. For instance, in making a graph to show the changes in the temperature of a room from day to day, there is no reason why the temperature should rise or fall from Wednesday to Thursday just because it is Wednesday or Thursday. To know the temperature on Thursday and Saturday is no help towards knowing it on Friday. But if we show the connection between months of the year and the average temperature for each month measured over, say, twenty years, a certain kinship between the quantities would reveal itself in the graph. We should find that the temperature was lowest at one time of the year and highest at another, and that we could tell approximately the average temperature on July 15th during the last twenty years if we knew the average temperature on July 1st and on August 1st. We shall give more examples of this kind later. Other examples of quantities such that the change in one in no way depends on the change in the other are—class attendance or marks and days of the week; imports or population and date of the year; distance walked and month in which walked. In all such cases the points found should simply be joined by straight lines, since any attempt to seek information for intermediate quantities must be futile. We know not whether, if one increases, the other will increase or decrease. Thus the usefulness of such graphs or charts is limited to the fact that they present a series of co-ordinated facts to the eye vividly and in such a form that they can be easily grasped. Most of the graphs printed in newspapers or used by children in a classroom are of this type. From an attendance or bad-mark or temperature chart, it is easy to report at a glance on the week's attendance or conduct or on the comfort of the room, to compare one week with another or one day with another. Such charts can be of real value both to teacher and class. The other day we saw in a newspaper a chart showing the rate of exchange in a neutral country for British, French, Italian, Russian and German money, from month to month since the beginning of the war. It was impossible to deduce from this what would happen in the future, or to read from it the rate of exchange on any given date, but at one glance the eye could for any time during the duration of the war appreciate its success or failure, from the point of view of any country concerned, and could note the balance between the various countries. No mere list of figures could (in any length of time) have been so assimilated as to give an equal amount of general information. The use of charts of this type to record economic, social and national results of importance is constantly on the increase; our pupils should be able to understand results recorded in this way.

(*b*) We shall now indicate examples of another type of connection, which afford the greatest contrast to the above instances. This is the case where one quantity depends definitely on the other, where the relation between them is so close that if one is known the other can always be calculated. Our children's acquaintance with such relationships must probably be limited in the Elementary School to direct and inverse proportion, although the relationships between numbers and their squares or square roots could also be easily developed. Mathematicians know, of course, of an endless variety of such relations and express them algebraically by equations connecting the distances along the two axes. For instance, if x measures the distance along OX, y the distance along OY, y always $= kx$ for direct proportion where k is some constant depending on the particular pair of quantities: $y = \frac{k}{x}$ for inverse proportion: $y = x^2$ or $\sqrt{x}$ for square or square root. If we recollect the types $y = ax + b$, $y = ax^2 + bx + c$, $x^2 + y^2 = a^2$, $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, we remember that in each type, when a certain number of points have been "plotted," they are found to lie, not in the haphazard way noticed in the previous cases, but always on definite curves or lines. In fact, the form of the curve is in each case the geometrical or graphical expression of the nature of the relationship existing between the two sets of changing quantities. In other words, the graph tells us such facts about the quantities as were indicated in (*a*), but also tells us about the law of their relationship; to use our previous metaphor, the acquaintances of the first case, having become relatives in this case, the graph will tell us whether the relationship is a filial one, a fraternal one, or an avuncular one !

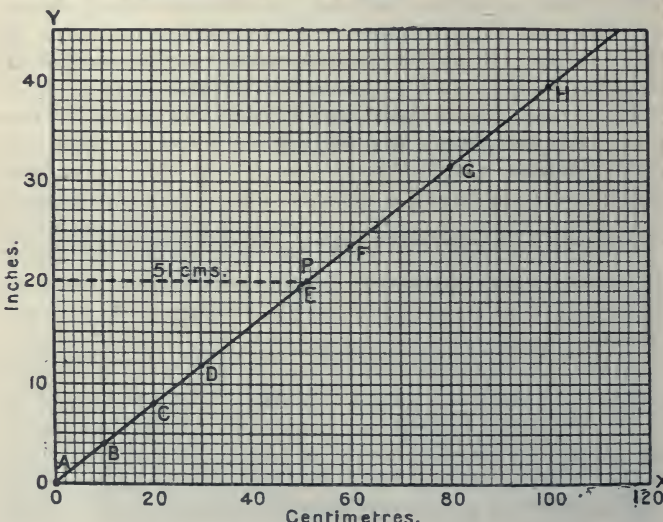
Let us study more closely the two common types of direct and inverse proportion. Take this example for direct proportion.

> "1 metre = 39·4 in., to nearest tenth of an inch. Make a graph to show the relation between inches and centimetres."

Let us first make a table of the quantities as they change, calculating by proportion—

cms.	100	0	10	20	30	50	60	80
inches	39·4	0	3·9	7·9	11·8	19·7	23·6	31·5

Now plot these points on the graph—

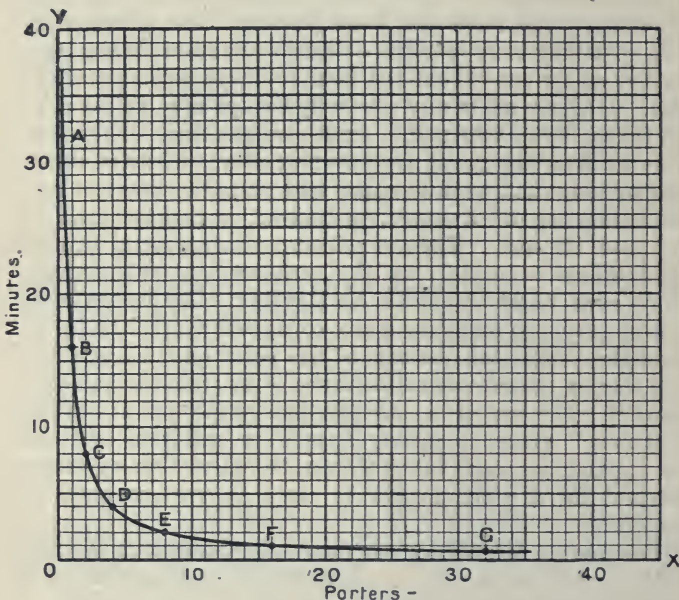


It will be seen that the points lie on a straight line. Children can try to explain why this is so, but a complete proof is beyond most children in Elementary Schools. After taking several examples of the direct proportion relationship, calculating the table and then plotting their results, they will be willing to believe the teacher's assertion that mathematicians can prove that the points always lie on a straight line. If this is so, in any case only two pairs of values need be found, since two points fix a straight line. Thus it is easy to draw the graph for any pair of quantities in direct proportion, such as kilograms and pounds, diameter and circumference of a circle, weight of a certain kind of tea and its cost. Moreover, intermediate results can be read off as accurately as the measurement will permit, since each point would lie on the line. For instance, in above graph, to find how many centimetres equal 20 in. we read the horizontal distance of the point on the line 20 in. up, that is the horizontal distance of P, which is 51 centimetres. The use of a straight line graph as a "ready reckoner" makes a strong appeal to children, and can be applied, amongst other things, to calculation of simple interest for a given length of time on various sums of money.

For inverse proportion we may take the case in Chapter XII, p. 177, of 4 porters emptying a luggage van in 4 minutes, and study the relation between the number of porters employed and the time taken. Let us calculate as before for this table—

Porters . .	4	8	12	16	32	3	2	1	$\frac{1}{2}$
Minutes . .	4	2	$1\frac{1}{3}$	1	$\frac{1}{2}$	$5\frac{1}{3}$	8	16	32

Here the form of the curve is harder to see. But when several examples of inverse proportion have been similarly treated, children see that the points found lie always in the same way with regard to the two axes. The form can then be discussed. It is, of course, possible to take intermediate readings in such a case, but the curve is so much more difficult to draw than a straight line that it cannot be used so universally as a "ready reckoner." Its main value with our children is to give emphasis to the distinction between direct and inverse proportion.



We can study, on above plan, the relationship between a number and its square or a number and its square root. In each case intermediate results can be obtained from the curve within limits fixed by the accuracy of our measurements.

(*c*) Between mere charts and graphs representing definite mathematical relationships lies another type of graph, where one of the quantities depends on the other, but in such a way that it is impossible to say exactly what is the law of their relationship. Very often this is because a third element enters in. For instance, the amount of an insurance premium depends not only on the number of annual payments made, but also on the results revealed by death statistics as to how many men per thousand of a certain age die at that age. Thus the relationship between premium and age is not a simple mathematical relationship, but that complicated by a question of vital statistics. Evidently if a man sells a series of saucepans of the same material but of differing diameters, the price will depend on the diameters. But it will depend also on the difficulty of making smaller things. In all such cases calculation is of no value, but when we are given a series of co-ordinated facts (see p. 233 for Age and Insurance Premium) and fix the points representing them, it is near the truth, but not the whole truth, to draw a curve through those points to represent the unknown relationship. We can use such curves to summarise information, but not to give additional information as in cases where the curve denoted a mathematical relationship. Any intermediate result, however, read from such a curve, will *probably* be approximately true, if we assume that the unknown law of the relationship continues to hold good. Illustrations of this can be best taken from work in Science or Geography: *e. g.* the time of the day and the length of a shadow; the sun's altitude and the time of the day; the amount of salt in a solution and its boiling-point. Often the similarity of curves drawn in this way to a curve representing some known mathematical law has led to scientific discoveries of importance.

The ground to be covered by children in Elementary Schools has incidentally been indicated. Charts of the first type, dealing with topics interesting to children, probably form the best introduction. Children can keep graphs of their own work in various subjects. Any good Arithmetic book gives examples suitable for practice. At first children should draw vertical lines instead of marking merely the *points* at the top of each line, and at first, too, it is good to choose quantities easily represented by lines: *e. g.* height or distance. But if a newspaper graph, or some interesting chart made by the teacher, is hung up on the wall a few days before the formal study of this work is undertaken, pupils often gather a surprising amount of information for themselves!

CHAPTER XIX

ALGEBRA IN ELEMENTARY SCHOOLS

The value of Algebra to human life being much less obvious than the value of Geometry, teachers have found some difficulty in deciding on the correct position of Algebra in the curricula of various types of schools. We shall here confine ourselves to a definite question: "To what extent and on what lines should Algebra be taught to children whose school education ends at fourteen years of age?" To help us to answer this question, we shall first try to form some estimate of the value of Algebra to human life and then glance for a moment at the history of its development in the race.

The value of Algebra to mathematicians is that on a sound knowledge of Algebra, progress in every branch of modern Mathematics depends; it is as essential for the future student of Mathematics to learn Algebra as it is for the ordinary person to learn to read. But almost every pupil of an Elementary School, who will later study Mathematics on these lines, is transferred to a Secondary School at eleven or twelve years of age. Hence this aspect of the value of Algebra may here be neglected by us.

Algebra has frequently been included in a curriculum because it was supposed to develop the minds of those studying it. For some consideration of the question of the place of mental training in a course of Elementary Mathematics, we refer the reader to Part I, Chapter IV. Here we simply ask two questions. To what extent did we ourselves derive mental training from learning the beginnings of Algebra on the usual mechanical lines? We do not deny that the mastery of a new mechanism gave many of us real pleasure. Are the reasoning powers of children under fourteen sufficiently developed to enable them to understand a *sound* treatment of Algebra as far as the development of rules of signs for positive and negative quantities? Granted that our pupils can mechanically apply these rules when known, are they fit to understand them or merely willing to accept them?

Algebra has a value to "practical" men in the great majority of occupations because it embodies useful generalisations from experience made by their predecessors, in condensed forms easily carried and readily applied. We refer here, of course, to the use of formulæ. Take a very simple case. A timber merchant has discovered that the quickest way of approximately calculating the surface of the sides of a rectangular box is to add its length to its breadth, double the result and multiply this by the height. After working out the surface area time after time in specific instances, timber merchants "generalised" from this experience and handed on their advice to their successors. But advice handed on in words, as indicated above, is slow to read and not too easy to apply. When, however, it has been expressed succinctly in the form $A = 2h(l + b)$, no man could desire a piece of advice more easily read and more simply applied by any one who has learnt to "read" simple Algebraic symbolism. Thus to "practical" men who wish to hand on to others their generalisations from experience or who wish to profit by others' advice, ability to make generalisations and express them in formulæ, and ability to read and apply a formula are essentials. Therefore, simple work on the above lines should, if possible, be included in the Elementary School curriculum. The only doubtful point is how far we shall carry the development of the symbolism, a point to which we shall return later. The concluding portion of the last paragraph has a bearing on this. Before passing on, we note that women even in their homes should find scope for their ability to make and apply useful formulæ. Confining ourselves to the ordinary Algebraic symbolism, we have the formula (similar to the wood-area one above) for calculating the number of rolls of paper (12 yds. by 21 in.) to purchase for a room: $R = \frac{2h(b + l)}{9 \times 7}$ if measurements of room are taken in feet.

Or to calculate the price of, say, $7\frac{1}{2}$ lb. of meat at $13\frac{1}{2}d.$ per lb., $C = ld + \frac{1}{2}(l + d) + \frac{1}{4}$, where $l =$ whole number of lb., $d =$ whole number of pence, $C =$ cost in pence. In addition to this, will not girls who have been trained in school to make and apply formulæ, using ordinary Algebraic symbolism, be better able to "take off" knitting and crochet patterns from books, since these patterns are simply formulæ using a peculiar knitting or crochet symbolism? Might not women develop formulæ for needlework, knitting, crochet and cooking which would give a complete generalisation of the way to make articles of a certain shape or design and foods of a certain taste and consistency? The limitation of the work done so far is that a new formula is required for every new size of the same article or food. This, however, is merely a suggestion to emphasise the point that Algebra has a value for "practical" women as well as for "practical" men, and that probably the last word has not yet been said in many directions which affect women's work.

The one value of Algebra which appeals to many who have studied its mysteries is that it enables us to answer puzzle problems which we cannot answer by Arithmetic, or to answer them with less labour than an Arithmetical solution demands. The reasoning necessary to answer many a problem, when expressed fully in words according to the laws of language (that is, in "rhetorical" form), is extremely hard to follow; the reader can test this by writing out in ordinary English style, without any contractions, the reasoning adopted in the solution of what is called a "problem involving a simple equation!" It shortens work and simplifies reasoning, to use contractions for the words, and perhaps a few odd symbols for operations (such as $\times$ and $+$), since the eye rather than the mind is overstrained. Such a form is called a "syncopated" form. But when one goes further and represents the unknown quantity by a symbolic letter, all the operations and quantities by symbols, abandoning altogether the attempt to write English, the reasoning such a problem demands becomes so obvious that quickly one learns to go through it mechanically or, in other words, to "solve the equation."

Thus the same reasoning may be expressed in three forms—rhetorical, syncopated and symbolic; the advantage in speed, and in economy of hand and brain effort, lies entirely with the last. Ability to solve an equation, however, is not only valuable in tackling puzzles and problems; it also enables one to use a formula made for one purpose for a different purpose. For instance: A woman has 6 rolls of standard English paper; her room is 15 ft. long and 12 ft. wide, and she intends to paper to a certain height and distemper the rest. How high must she paper?

$$R = \frac{2h(b+l)}{9 \times 7}$$

Substitute

$$6 = \frac{2h(15+12)}{9 \times 7} = \frac{2h \times \overset{3}{\cancel{27}}}{\cancel{9} \times 7}$$

$$6 = \frac{6h}{7}$$

$$\therefore h = 7.$$

Thus a formula made to calculate amount of paper has been used to calculate a height. In advanced Mathematics men often struggle to solve an equation in order that, from one formula known to be true, another can be deduced which will be true and will prove itself true when tried. An engineering professor was approached by his workmen because some engine was out of order. He performed some mathematical calculations with a formula known to be true, and, expressing the result of his calculation in words, he gave advice to the workmen. They carried out his advice, found the engine working correctly, and admiringly said, "The Professor can see through three inches of solid steel." He could not do that, but he could solve what was probably a troublesome equation! Hence ability to solve equations means power to the "practical" man and woman, and should be developed, if possible, in the Elementary School. The doubtful point again is simply: "To what extent will young pupils' imperfect reasoning powers allow this ability to be developed?" Fortunately the simple equation is a very useful type.

The development of Algebra in the race was a slow one, probably retarded unduly by the fact that, in Greek, letters of the alphabet referred to definite numbers; hence symbols for number in general were not ready to hand. The beginnings of symbolism for operations are found in the papyrus of Ahmes, about 2000 B.C. He used a pair of legs walking forward for addition, walking backwards for subtraction! Our + and — are supposed to have been used by Italian merchants in the Middle Ages as marks on sacks which were over weight or under weight; mathematicians adopted them from the merchants. But further developments of symbolism were slow in coming. The beginnings of Algebra were attempts to solve problems, and at times problems leading to indeterminate equations. Diophantus, a Greek of Alexandria, in the fourth century A.D., was particularly interested in such problems, but his solutions were put forward in the syncopated form; symbolism was still in its infancy. The Hindus and Arabs within the first twelve centuries of the present era made great advances, but in a purely rhetorical form. One Arab, called Alkarismi (*c.* A.D. 800) named his work "Algebr," which means "restoration." It refers to the fact that, in solving an equation, the same magnitude must be added to or subtracted from both sides; from this source comes our word Algebra. But although these Hindus and Arabs did much work in Algebra, and arrived at many of our laws of combination and methods of solving equations, their work was sadly handicapped for want of a good symbolism. The mental effort required for rhetorical and syncopated Algebra was tremendous! It was not until the close of the sixteenth century that an Italian called Vieta introduced letters as symbols for numbers in general. Within the next hundred years much progress was made; Napier invented logarithms, while Descartes introduced the index notation and was the first to give an interpretation of negative quantities. Thus the great advance which in Geometry was made between 600 and 200 B.C., in Algebra was delayed until 1500–1650. Then Algebra came into its own as the foundation of all advanced work in Mathematics; while with the beginnings of scientific discovery in the seventeenth and eighteenth centuries Algebraic symbolism and the Algebraic formula as a method of recording important results began to have a value for "practical" men.

Regarding, then, the stage of mental development of the pupil who leaves school at fourteen, the value of Algebra in human life and its slow development in the race, we suggest the following treatment of Algebra as suitable for an Elementary School curriculum.

1. No treatment of positive and negative quantities should, under ordinary conditions, be attempted in an Elementary School. $+$ and $-$ should be regarded as signs for the operations of addition and subtraction. Hence such statements as $-(3-4)=-3+4$, or $-4\times-3=+12$, or $-3x=12$, and $\therefore x=-4$, have no place in our syllabuses.

2. Our pupils should learn to generalise from their Arithmetical results, to express their generalisations in Algebraic symbolism and to use the formulæ so made to save labour. They should also learn to read and use formulæ made by other people who know more than they know. But the symbolism introduced should exclude signs for positive and negative quantities as such, and thus must limit itself to positive integral indices; even these should not be introduced too quickly. This work leads to much of the ordinary manipulation of Algebraic symbols now done in many schools.

3. Our pupils should learn how to solve problems, including proportion problems, by solving a simple equation, provided that its solution is not a negative quantity. They should apply this ability to answering "inverse" questions on formulæ used in Arithmetic and science, so extending their power over a formula.

(1) requires no further comment, but a few illustrations of the way in which (2) and (3) can be developed may prove useful to those who are unaccustomed to the exclusion of negative quantities.

Examples of Generalisation and Use of a Formula.

The variety of possibilities is very great; these are merely typical examples.

1. Little children in studying a square can be asked to find the distance round it; they learn to measure one side and multiply its length by 4. After measuring a few squares they can try to express in words what they have done—*i. e.* try to form a generalisation to help people who do not know what they know. They can shorten their generalisation to

$$\text{Distance round} = \text{side} \times 4$$

and shorten it still further to

$$D = s \times 4 \text{ and so to } D = 4s$$

learning that $4s$ is not 4 tens and s units (as in 42), but means 4 times s in this new language!

Obviously, by similar steps, children can write down the total distance round two or four or seven squares of equal sides or of unequal sides; subtraction can be introduced, provided the answer is not negative. As soon as they know that the area of a square is got by multiplying a side by a side, they can make the formula $A = ss$, and if it seems good (which is doubtful at such an early stage) write this as $A = s^2$. This explains why the index *2* is always called *square* instead of *two*.

2. Children who have found out the sum of the angles of a triangle can be asked to find the base angles of an isosceles triangle whose vertical angle is, say, 40°. This leaves 180° — 40° for the two base angles and so 70° for each. After a few definite questions, they can be asked to state their method in words, that is, to make a generalisation. "Subtract the vertical angle from 180° and halve your answer" can be shortened to

$$\text{base angle} = \text{half of } (180 - \text{vertical angle})$$

(the bracket is here useful); and so to

$$b = \tfrac{1}{2}(180 - v) \text{ or } \frac{180 - v}{2}$$

They can answer questions on finding the base angle from the formula.

3. All the results for finding areas and volumes of various mathematical figures and solids should be embodied

in formulæ. This work gives valuable practice in making and using a formula.

4. After working several questions on finding Simple Interest, children can explain how they worked them and so pass to the well-known formula—

$$I = \frac{PRT}{100}$$

5. To develop the Algebraic law for distributing a product, children can begin with this example—

There are 40 children in Standard IV. and 35 in Standard V. Each child has 3 notebooks and 5 printed books. Write down the total number of books in the classes.

This is 75×8 or $(40 + 35)(3 + 5)$ or 600.

Ask them now to write down all the Standard IV. books and all the Standard V. books separately and lead thus to $(40 + 35)\ (3 + 5) = 40\ (3 + 5) + 35\ (3 + 5)$; then to write down notebooks in each class as separate from printed books, and lead thus to $(40 + 35)(3 + 5) = 40 \times 3 + 40 \times 5 + 35 \times 3 + 35 \times 5$.

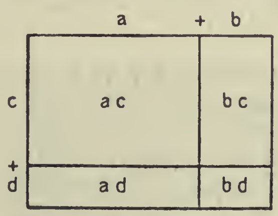
The meaning of each term in the product should be made clear.

After several examples this leads to—

a children in St. IV. b children in St. V.
c notebooks d printed books

$$\text{Total number of books} = (a + b)(c + d) = ac + ad + bc + bd.$$

This can be well illustrated by areas of rectangles.



From this follow the formulæ for $(a + b)^2$, $(a - b)^2$, $a^2 - b^2$, all of which can be applied to shortening calculations and illustrated by diagrams.

The above work can obviously be extended to introduce most of the ground usually covered in Elementary Schools. But children will use "symbolical expression" early as a short way of explaining to others how to do sums they themselves know how to do, while "substitution" will be practised on formulæ made by the children to shorten their future labour, or made by others to give the children new information. Children might sometimes be able to bring such formulæ to school from their fathers; the advantages of this are evident.

Solution of Simple Equations and their Application to Problems and Formulæ.

1. The rules are best developed from easy problems worked mentally.

e. g. (i) I think of a number. I have added 9 to it. The result is 65. What is the number?

After a few of this type, the reasoning can be set down symbolically—

$$\begin{aligned} n + 9 &= 65 \\ \therefore n &= 65 - 9 \\ n &= 56 \end{aligned}$$

I think of a number. I have added to it 9 then 4, then 11. The result is 65. What is the number?

This leads to

$$\begin{aligned} n + 9 + 4 + 11 &= 65 \\ n + 24 &= 65 \\ n &= 65 - 24 \\ n &= 21 \\ \text{or } n &= 65 - 9 - 4 - 11 \\ n &= 21 \end{aligned}$$

(ii) I think of a number; I subtract 7 from it. The result is 15. What is the number?

I think of a number; I subtract 7 from it and add 12. The result is 17. What is the number?

$$\begin{aligned} n - 7 &= 15 \\ \therefore\ n &= 15 + 7 \\ n &= 22 \end{aligned} \qquad \begin{aligned} n - 7 + 12 &= 17 \\ n + 5 &= 17 \\ n &= 17 - 5 \\ n &= 12 \end{aligned}$$

(iii) 4 times a certain number is 36. What is the number?

This is very easy—

$$\begin{aligned} 4n &= 36 \\ n &= 9 \end{aligned}$$

(iv) 4 times a certain number with 9 added on is 57. What is the number?

$$\begin{aligned} 4n + 9 &= 57 \\ 4n &= 57 - 9 \\ 4n &= 48 \\ n &= 12 \end{aligned}$$

(v) I think of a number and divide it by 5. My answer is 4. What is the number?

$$\begin{aligned} \frac{n}{5} &= 4 \\ n &= 20 \end{aligned}$$

From these and similar examples it is easy after a time to deduce the usual rules—

(i) If a quantity disappears from one side it appears on the other with its sign changed.

(ii) Both sides of an equation can be multiplied or divided by the same number without spoiling the balance of the equation.

These results have been obtained without any consideration of negative quantities and they should not be extended to negative quantities without full discussion.

2. The application of simple equations to problems needs no comment. Obviously this is the stage at which the ratio method of answering proportion problems can be introduced. See Part II., Chapter XII.

3. As examples of the application of simple equations to formulæ, let us use the formulæ mentioned earlier in the chapter.

e.g. (i) An isosceles triangle has its base angles each 40°. Find its vertical angle from the formula

$$b = \frac{180 - v}{2}$$

$$40 = \frac{180 - v}{2}$$

$$\therefore\ 80 = 180 - v$$

$$\therefore\ v = 180 - 80$$

$$v = 100.$$

(ii) The S. I. on £233 6*s.* 8*d.* for 3 years is £65. Find the rate % per annum from the formula $I = \frac{PRT}{100}$.

$$65 = \frac{233\frac{1}{3} \times R \times 3}{100}$$

$$\therefore\ 6500 = \frac{700}{\cancel{3}} \times \cancel{3} \times R$$

$$\therefore\ R = \frac{65\cancel{00}}{7\cancel{00}} = 9\tfrac{2}{7}$$

PART III

CHAPTER I

THE PROBLEM OF MOTIVE

BEHIND every action performed by man, woman, or child, lies some motive prompting the performance of that action. Although the work to be done may be fixed in type, yet the varying nature of the prompting force affects the quality of the results, gives colour, as it were, to the action. Thus any person concerned in the production of work by other persons must meet this problem of motive. We shall first glance, for the sake of illustration, at the national duties now being performed by many citizens, and then pass to a consideration of the motives which prompt our pupils to learn Arithmetic.

Three years ago, the great majority of the people of this country was unaware of any call to perform service of national importance; now every one knows that to the extent of his ability, duty summons him to help his country. Moreover, the State has the power of forcing reluctant individuals to perform such work. So arises a great cleavage in the nation. Some serve voluntarily in whatever way may be possible; others serve only under compulsion. We all recognise that a job done by one man under compulsion and by another man voluntarily, may be externally the same and yet in spirit radically different; we all recognise, too, the superiority of voluntary work. As each new call comes for soldiers and sailors, for munition workers, for money, for economy in food, the attempt is made to secure as large a supply as possible of willing service; only after this supply is exhausted does the State resort to compulsion. We see the waste of effort involved in any system of compulsion, and we desire the colour and fire and inspiration which accompany good voluntary work, but which cannot accompany labour until it has ceased to be forced and has become willing service. These are truisms in matters of national importance.

With these ideas in our minds let us approach the question of how we are to stir up our children to do Arithmetic. Probably many teachers, if they voiced their private opinions on this problem, would summarise them after this fashion: three-quarters of our pupils do Arithmetic only because they must, one quarter of them do Arithmetic because they naturally like working with numbers and are good at it. Such a view, though becoming less and less prevalent, is still held by not a few. If a personal reminiscence may be pardoned, the author well remembers her first experience of Arithmetic in an Elementary School. She was sent as a student to teach Arithmetic to Standard VII., a class of sixty taught in a large room and separated from another large class by a blackboard screen. To reach this room she had to pass every day through a room occupied by quite little children busy at Arithmetic; she never passed through that room without seeing children in tears. She was puzzled during her first lesson by the sounds which came from the other end of the room. It seemed as if notebooks were continually being thrown on the desk. Next day she discovered that the rule with the other class was that each sum wrong earned one stroke from the "tawse," the Scotch substitute for a cane! Of course, such a state of matters is now much less frequent, but do not most of us know of some school where circumstances comparable with these exist, where the attitude is that most children will do Arithmetic only when forced to do it, and that a cane is the simplest instrument for the application of force? No regulations with regard to corporal punishment imposed on schools by outside authorities will ever reach the root of the evil. To forbid such teachers to punish and at the same time to insist on good results from large classes in crowded rooms, seems like putting an undue strain on a teacher's honesty and his loyalty to the authorities. In the great majority of such cases the teacher himself would rather produce good results by happier means, but he knows not how to succeed. And when occasionally a teacher believing in other methods than compulsion, struggles hard in such schools to achieve results from children used to the compulsory régime, are his efforts not often to some extent frustrated by a head who comes in occasionally to cane every child whose sums are wrong? When a teacher is suspected in such schools of not exercising sufficient compulsion, the head steps in to "put things right." This old régime is dying, but it is not yet dead. Behind it is the wrong idea, the idea that most children will do Arithmetic only because they must; that none save bright children, good at figuring, can enjoy Arithmetic.

Let us now pass to a more congenial topic. Many teachers and many schools run the teaching of Arithmetic on totally different lines and yet secure good results. How is this achieved? The secret appears to be that we often fail to realise the reinforcements we teachers can receive from our children. In every human child are certain natural desires which prompt their energetic actions outside school. Is it impossible to yoke these desires to the service of Arithmetic? We shall look at a few of them in detail and note how we can utilise them for our educational purposes.

First, nearly every child loves a puzzle, enjoys discovering how to accomplish a piece of work never performed before. If only we could make Arithmetic lessons more like puzzle-solving games, part of our difficulty would be overcome. But there are hindrances in the way. Puzzles become sums, the game becomes a task, when failure invariably means punishment or censure, or when the aim is mere preparation for an examination. Another difficulty is that much of the new work children cannot find out for themselves: work demonstrated cannot arouse the puzzle instinct. Then, too, in working lessons on familiar topics, only the brighter pupils are able to tackle problems for themselves. The other children must be told what to do, and to be told how to solve a puzzle spoils the game. The answer to the first point is that with children accustomed to the régime of fear, it will take time to produce a happier atmosphere, but that when this atmosphere has been produced, the puzzle-loving instinct will be a magnificent aid to us in teaching Arithmetic. It is not of much use with the majority of a class until fresh air and sunshine have replaced the fogs of fear and the mists of suspicion. The answer to the second difficulty is that children truly can find out how to do new work in Arithmetic if the work be subdivided into easy stages, as has been continually suggested in Part II.; that even those weaker children who do not discover the new for themselves, when they have once tried to discover it, are more interested in the teacher's demonstration, or, preferably, in their friends' solutions. In other words, when we "waste time" in letting children attempt new work by themselves, we are replacing forced labour by voluntary effort, and the time wasted for a moment will be recovered when our pupils are once into the steady pace of the work. The answer to the third point can be fully taken only in the succeeding chapter: the problem is really the problem of providing harder puzzles for quick children, easier ones for children of mediocre ability, and very simple ones for the weaklings of our classes. No one can be expected to go on loving puzzles if they never come out in his hand. If children, too, acquire this attitude to Arithmetic, they may be more safely entrusted with answers than we often suspect, even trusted to use an answer to help them to do a difficult sum. "Copying" would be practically non-existent; but we shall return to this matter a little later. In conclusion, we may note that the most frequent reason for a bright pupil's love of Arithmetic is the satisfaction of his natural desire to solve a puzzle or make some fresh discovery.

Another natural desire in the child which is really helpful to us in Arithmetic is his love of playing at being a "grown-up." For children all that pertains to adult life has a certain romantic attraction: dolls are mothered, schools are taught, soldiers are drilled, the sick are given medicine, operations are performed, marriages and funerals are held with equal enjoyment. The child is vastly interested in *life*, although he is frequently uninterested in school. We have only to remember the games we loved the best when we were children, our passionate desire to be considered really grown-up, our wonderings as to our future careers, to realise that anything that can be shown to have to do with adult life has a charm of its own for every child. But in the earlier parts of this book we have seen how important a part Arithmetic plays in the doings of adults, how Arithmetic can be so taught that it is kept in touch with human life. If, in beginning each new topic, we take pains to give our pupils an inkling of its value for life, and if, in the few cases where this is impossible, we ask them to accept our word that the new work is valuable, and later take time to indicate its value, we shall find that a sense of the "worth-whileness" of Arithmetic will be developed even in our weaker pupils and will stimulate all to greater effort. Here is a motive for willing toil most potent in its working, most safe in its effects on children. It cannot too often be repeated that Arithmetic is not for examinations or inspections, but for *life*.

If our pupils had been taught all their Arithmetic by methods which satisfy the natural human desires we have just indicated, probably little else in the way of motive would be necessary. Occasionally, from time to time, with awkward individuals or with ordinary individuals under awkward circumstances, voluntary effort would for a brief time be replaced by forced labour. The wiser the teacher, the wider the range of his understanding and sympathy and the surer his touch on human weakness and misery, the less frequently will he require to resort to compulsion. He will regard it always as something to be regretted, though he will admit that from time to time it was the best possible under the circumstances. If he can develop some method of allowing the class as a whole to decide when compulsion is necessary and how force is to be applied, he will secure the greatest possible effect from its use. After all, even sensible adults resent compulsion by superior beings much more than compulsion by their peers!

But, unfortunately, we often begin work in the middle of a muddle! At its best it may be that we have a new class previously taught by some one not very good at yoking children's natural desires to the service of Arithmetic; at its worst it may be a class in a fresh school where the Arithmetic teaching is bad, where compulsion is the order of the day, where the children regard their teachers as indifferent beings or as enemies, but never as friends. To refuse to apply general compulsion means bad results, as the children mistake kindness and absence of harshness for softness and weakness. It takes time for firmness, justice, and good teaching on interesting lines to convince children that the teacher wants to be friendly and to help, but yet means to have work; prefers voluntary effort, but yet, if need be, will use compulsion; punishes because of children's faults and not through harshness or caprice. Are there any other natural desires in our children which we may use under such circumstances to secure voluntary effort and reduce forced labour to a minimum?

Many of our pupils have a strong desire to excel; of this much use is made in schools. But to stir up effort in pure competition, in the desire to be the best in the class, has very serious drawbacks. It affects only that part of the class whose progress is the least part of our problem, while we have no guarantee that the selfish motive behind the effort put forth will not have a pernicious effect on our pupils' moral growth.

What is probably most useful under such circumstances is a modified use of this desire to excel. To do better for the pleasure of surpassing our own previous best, to do better for the sake of some body of people to which we belong, even to do better for the sake of pleasing some individual or of obtaining some desirable reward—these are all safer and more moral motives than to do better simply to defeat other people. Perhaps the most useful step in advance which a teacher can make with such a difficult class, is to invent some method of marking in which effort counts for more than success, laziness for more than failure. We suggest two methods which have proved useful. One is to have a large sheet on the classroom wall with all the children's names on it, and with places opposite for entering good and bad marks. With little children stars of gummed paper in two colours can be used, the affixing of good or bad stars being made a ceremonial occasion; with older children crosses in two colours of crayon will suffice. The idea is that when the teacher comes across work better than the individual's usual, work with effort in it, a good mark be made; when he comes across work that is slack or lazy, a bad mark, unless there is some adequate excuse. The children can see their progress and other pupils' progress, while the progress of the class as a whole in effort from week to week can be shown on a chart by counting good and bad marks and recording the difference. In addition, the teacher has a most useful aid in tackling the thoroughly lazy individual. If punishment be promised, say, after five bad marks in one week, no child can feel that punishment is not merited by his laziness. Of course, the system necessitates a sensible teacher in whom the class have confidence, and it should never be required for long. But in the first months with a new class in a difficult situation, it may be helpful.

The other plan is to give marks for work on the usual lines adopted in the school, but to add a + for every piece of work showing care and effort, a — for every piece of work showing carelessness and slackness. The teacher can keep a record, and so can each child; the promise of some little reward for those who make a certain amount of advance in a month or a term may help to secure an increasing amount of voluntary effort. The real problem is, of course, to get over the initial period while the class becomes acquainted with a new kind of teacher.

One of the motives indicated above, the motive of doing better for the sake of some group to which we belong, can be made general use of in almost any circumstances. The class can be divided into two or more teams under captains, which "play" each other in Arithmetic for some convenient period. Public spirit can be developed by such means, and the time spent in choosing teams and in comparing their results is saved by securing keener effort. Often, too, pressure can be brought to bear on "slackers" more effectually by captains than by teachers!

One troublesome characteristic of many classes is closely connected with the problem we have been considering: we refer to the prevalence of what is known as "copying," to the difficulty of securing independent work from the majority of our pupils. To meet this difficulty most teachers say that it is absolutely necessary to provide different work for children sitting side by side. However desirable it may be to do this when there are special reasons for testing children's powers, to make a constant practice of it obviously uses in a wrong direction the teacher's energy and preparation. The class cannot so easily be worked in subdivisions according to ability, if for each subdivision double work must be set. In the next chapter we shall discuss in greater detail this aspect of the copying problem. At present let us try to see the matter as children see it and as the problem of motive affects it. A better way of escape should be possible, since in some schools and some classes the difficulty as a class-problem simply does not exist, although from time to time it emerges in individual cases.

We believe that the child's attitude to this matter may be simply put in this way: "We are encouraged to help our neighbours in other matters, and we like helping our neighbours and being helped by them in other matters: can it be wrong to help each other over Arithmetic? Of course, teacher says we are not to; but he is cross if our sums are wrong even when we try our best. At any rate, we need not trouble as long as he doesn't know that we are helping each other." Needless to say, this is not the real attitude in every case of copying: lazy children will do it simply to save themselves effort. But in such cases we have still to understand why the other children help the lazy ones.

An analysis of the child's point of view brings out the three main features of the situation. Children fail to understand why individual work is demanded. Children regard the teacher's order as having no *moral* authority because of his unjust demand for correct work at all costs. Children, being in league against a teacher whom they distrust, pass from the broad paths on which the sun shines, to the narrow tracks of dishonesty and deceit. In fact, copying is just exactly what we should expect from a class, the majority of whom give only forced labour. For the teacher who is unable to see the possibility of any general motive for work except fear of consequences, will be equally unable to make his pupils understand why individual work is sometimes necessary. The production of good results by force leads to children distrusting the teacher as a moral guide: they know he is often wrong, and so disobedience is sanctioned by the child's conscience. The more harsh the application of force, the more easily will children's feet be turned into the paths of dishonesty and deceit.

On the other hand, if the class has been trained to put forth much voluntary effort through the motives suggested above, knowing that Arithmetic has to do with life and is therefore a matter for the individual, they will see sense in being independent of help and will be slow to take unnecessary assistance; regarding Arithmetic as a puzzle-solving game, they will be eager to find things out for themselves; being on friendly terms with their teacher and being set free from any undue fear of punishment for poor work, they will have no moral sanction for disobedience and no strong inducement to dishonesty. Obviously copying would be a rare occurrence in such a class.

What, then, can be done to cure a class suffering from this disease? We do not call it a vice, because in classes where it is prevalent the children themselves do not see that it is really wrong. There is no public opinion against it: until there is, little headway can be made. First, it is generally wise frankly to talk things over with the children, showing that one appreciates their point of view, but trying to help them to a new point of view. It is right to help one's neighbour, and the wise teacher will make opportunities for such help, allowing equally able children to work together, or a quick, clear child to help a backward one, or occasionally forming groups to work under a captain; from such experience children quickly distinguish between wise and unwise assistance. But the teacher wants to help also: the children must be encouraged to ask for necessary help, and the others must see that the teacher cannot know how to help the class to do Arithmetic unless he has a chance of sometimes seeing just how much each individual can do. The children can be asked to suggest other reasons for sometimes working quite independently: if they can discover the reasons for themselves, public opinion will thereby be developed. They can perhaps for a time be allowed to decide for themselves on which days they will work independently. After a little experience of this they will be more ready to accept the teacher's right to order individual work. If the teacher feels it wise to set A and B work for a weekly or fortnightly test, the reason emphasised should be the awkwardness of the situation when one does by accident see an answer different from one's own in a test. By such means, within a few weeks' time, provided that one's efforts are not undone by undue punishment for poor work applied accidentally by oneself or deliberately by one's head, some kind of public opinion will exist. Now remains the difficulty of tackling the thoroughly lazy and the really deceitful. Where exigencies of space permit, undoubtedly the best cure is to set each of these individuals to work by himself in such a position that copying is impossible. The gregarious instinct comes to our assistance: no child likes continued isolation, and a week of such work would probably cure most children! When the room is unduly crowded and floor space and extra seats are mere dreams, one must compromise between the labour of setting separate work to these troublesome individuals and the application of compulsion. Unfortunately, compulsion of itself is seldom sufficient to cure this evil: there is no short, sure way in this case. But gradually, by such means as we have indicated above, the class as a whole does pass from dependence to independence, while the teacher's time and energy can be devoted to better purposes than the continual provision and correction of two sets of work similar in scope.

CHAPTER II

TEACHING THE AVERAGE CHILD

ONE problem which has met us from time to time throughout the earlier parts of this book must now be faced—the problem of teaching Arithmetic to large classes containing pupils of varying ability. We found, for instance, that the development of speed and accuracy in a class depends to a great extent on setting the correct amount of suitable work to every member of the class; we found that the development of ability to tackle problems depends largely on setting harder problems for the best members, easier ones for the mediocre, and babyish ones for the backward. We noted also that the attempt to teach a class as if it were entirely composed of samples of the "average child" results in a serious neglect of the weak pupils, and in a failure to make the best of the bright pupils. By some means or other we must, in our teaching, provide for the variety of individual knowledge and skill that presents itself before us. The problem varies in acuteness, however, from school to school. In a district with a floating population, a school large enough to allow of only one class for each standard finds the problem most harassing; in a settled district a school large enough to have two or more classes for each standard feels the problem least distressing. But in the majority of schools the problem exists, whether there be any organised attempt to solve it or not. Obviously it becomes increasingly acute as a class rises in school, unless some such organised attempt be made.

Let us first try to make a rough classification of the pupils who are before us in the average class. We can usually subdivide them roughly into three groups, which, for convenience, we may label A's, B's and C's. The A's are quick and alert and usually have good memories, though some may be superficial. They form, perhaps, 25% of an ordinary class; even as much as 50% of a good class may belong to this group, or as little as 10% of a weak class. Such children usually enjoy Arithmetic, because the puzzle element appeals to them and they have the frequent joy of success. They require little definite teaching, but an abundance of educative work. At the other end are the C's, dull, slow and often very inaccurate, probably badly grounded in Arithmetic or held back unduly by physical conditions. The real C child is seldom an active nuisance in class, but listlessness and inattention mark him as their prey. In the ordinary class of the average school there are probably more C's than A's, say 30%, rising to 45% of a weak class and falling to 10% of a good class. Between these two extremes sit the mediocre B's, good, useful members of the community, the best of them held back below the A's either by slow-moving minds or by inaccuracy, the worst of them raised above the C's by a greater amount of initiative and vigour. They form the largest section of the ordinary class, the percentage working out at 45% on the above reckoning for A's and C's. If we wish to be a little more discriminating in our judgment, we might pick out from the A's at the top of the class a very few children with real mathematical ability, who will forge steadily ahead under almost any teaching, provided only that their time be not wasted; from the C's at the bottom of the class, a very few with real intellectual deficiency, who will never be able to perform even the simplest abstract calculations without assistance. In districts which provide for the mentally deficient, such pupils will, of course, be practically non-existent in the ordinary classes. We thus have three main sections to play off against each other in such a way that our teaching may secure the greatest good to the class as a whole.

It is obviously impossible to give dogmatic advice for such a complex situation; we can only indicate some of the lines along which a solution of this problem may be found. The art of teaching a class Arithmetic may, from this point of view, be compared with the art of playing an organ with three or five stops. It is clearly bad always to pull out all the stops and play steadily on, just as it is clearly bad to attempt to teach Arithmetic to a class as if all its members were specimens of "the average child." The impossibility of giving to the budding organist advice on the manipulation of his stops, suited to every piece of music played and to his own individuality, is equalled by the impossibility of giving to the teacher advice on the handling of his class-sections which will be suited to every type of lesson and to his own physical, mental and moral qualifications. For both types of work an artist is required, and each artist must give his own rendering of the piece of work in question. With this idea in our minds, we pass on to indicate possibilities.

Some schools have found a partial solution of this problem by sending children up or down to other standards for Arithmetic lessons. This can be done, because in most schools Arithmetic lessons occur at the same hour. Only two objections to it are raised: it interferes with the marking of registers, and at a certain stage in the school Arithmetic lessons usually become ten minutes longer. But surely the very few late scholars arriving after 9.30 could be taught to report themselves to their own teacher before going elsewhere for Arithmetic; surely some time-table plan could be adopted whereby at the one awkward transition, children with a spare ten minutes could spend them to advantage, and those requiring an extra ten minutes for Arithmetic could avoid missing an essential lesson in their own class. The plan must, of course, work in two directions: one must receive the best from the standard below if one gives up one's best to the standard above; one must receive the worst from the standard above if one gives up one's worst to the standard below. The plan can be worked merely for the best of the A's and the worst of the C's, or some more thoroughgoing scheme can be developed.

Another way of dealing with the very few at the top and the bottom of the class is to provide the best, as soon as they have mastered the new work along with their standard, with our own old Arithmetic books with answers, or with cards and answers, and to provide the worst with a good Arithmetic book for a lower standard and with apparatus and tables. Both sets can receive only a little attention; fifteen minutes a week for each being often the maximum possible, but they will make some headway for themselves, and every step onwards will be clear gain. This, however, still leaves us with the ordinary A's, B's and C's.

One line along which a solution for these may be found is subdivision of the class, either permanently or, perhaps better, temporarily, into smaller sections. New work can be taken with the class as a whole, the quick children playing the part of explorers. This helps all, for the brighter children tend by themselves to work too quickly for thoroughness, while the slower children by themselves miss the stimulation of more eager minds. As the new becomes familiar, the sections emerge distinctly. The A's quickly master the new, and are ready for miscellaneous problems combining the new with the old, or for other new work; the B's do finally master the new, but they move much more slowly, the best of them gradually being transferred to work with the A's; the C's must have the new thoroughly taught for a second time, and are never able to work more than simple examples embodying it. From day to day we must transfer our main attention from section to section. The A's work can sometimes be marked outside class, sometimes marked by each individual from answers given, sometimes marked by the first children to finish correctly the day's work; or they can work in pairs and help each other. They will require occasional assistance with a hard problem, and a certain small amount of supervision; perhaps at a later stage they will be taught new work which the others do not learn. The B's work requires much attention to secure care and accuracy, while the C's only towards the later stages can be left for long to work alone. In general the time must be pretty evenly divided between B's and C's, either by giving one the most of the time one day and the other the next, or by dividing the attention in one lesson pretty equally. The

A's should always receive a small part of our time, though on many days five minutes will suffice. The plan sounds more complicated than it actually proves in practice. For one thing, work as a whole goes more smoothly when each child has questions within his ability, and can taste the joy of success and the greater joy of rising to a higher division. For another, it is easier and quicker to give satisfactory oral teaching to a small section whose difficulties are much alike than to a large class, part of which is always wearied by the others' troubles. It is possible, for instance, to take a small section round the blackboard. We can utilise, too, the best in each section to help the weaker ones or to assist us in marking. The setting of three sets of questions is much helped if there are two different books of examples for the standard, or if a second blackboard is provided, and if double work, to prevent copying, is not demanded. Often the three sets may be reduced to two, the same examples serving for B's and C's, or for B's and A's, if the easier ones are placed at the beginning. Occasionally it may be possible to arrange for one part of the class to do Composition or Silent-Reading with questions to answer from their reading, while the other part is taught Arithmetic : that is, possible to arrange for combining two periods of the day's work. This is particularly useful for practical Arithmetic lessons in large classes, a point to which we shall recur in a later chapter. We have already suggested a weekly lesson in pure drill work to develop speed and accuracy in the fundamental operations. If this be given to different sections on different days, we shall again be more free to give attention to the individual sections. All such suggestions, of course, apply more in upper classes than in Standards I. and II., but there is no reason why subdivision should not be begun at least in Standard II. with benefit to all concerned.

Another aid towards dealing with pupils of varying ability we find in grouping our problems. An example will best explain. Take, for instance, a Standard II., able to do compound addition and multiplication of money, extending them to length. We wish to teach our children to answer written questions involving the addition and multiplication of feet and inches, all knowing already how to measure objects in feet and inches. One may reasonably expect the better half of our class to be able to do the new for themselves. With this in our minds we go to the lesson, and put up on the blackboard these examples—

1. A joiner has 3 boards of the following lengths: 5 ft. 6 in., 7 ft. 10 in., 3 ft. 2 in. How many feet and inches long are all the boards put together?
2. John is 3 ft. 7 in. high, Tom is 2 ft. 11 in. high, Annie 2 ft. 8 in. high, Dorothy 3 ft. 1 in. high. If they lay down to make as long a line as possible, how long would it be?
3. A man needs 2 ft. 5 in. of string to tie up a parcel: how much string will be needed to tie up five parcels of the same size?
4. In a classroom are seven desks, each 5 ft. 6 in. long. What length of row would they make?

We perhaps do a little oral work, revising the changing of pence to shillings and the connection between a foot and an inch. We then read over the first two questions, remark on their newness, but express our expectation that some will get them right, and set the class on to try, telling them to stand when finished. If, in going round, we notice that two or three are forging far ahead of the others, we can write on the board in question (1) above the original figures, using coloured chalk, a different series of numbers, *e.g.* 17 ft. 9 in., 4 ft. 11 in., 6 ft. 5 in.; when these pupils stand up we can tell them to work the new question (1). This may not be necessary; that is, about a quarter of the class may finish (2) at much the same time. The whole class is then stopped, and question (1) is worked carefully on the blackboard. The common mistakes are to leave the inches without changing them to feet and to divide by 10 instead of 12. The extent to which these mistakes are made will determine whether we shall also work question (2) or merely give its answer.

For the second part of question (1) we need give only the answer. Obviously those who get either one sum or two right, here deserve praise because they have discovered the new, be their discovery quick or slow. The others have tried for themselves. The class can now be asked to work questions (3) and (4), and harder numbers may be put in one of them to keep any very quick minds at work. Question (3) is taken with class as soon as some have finished the two or three set; the others will be taken in full, or merely corrected by reading out their answers. If time remains in this first lesson, those with questions (1) to (4) right may work from their books; those with them wrong may correct their mistakes, the teacher probably not finding out the extent of their misunderstanding till the following lesson. This plan of setting two similar questions, correcting the first fully, but merely giving the answer to the second, is frequently most useful when new work is being taken, as a method of keeping quicker children busy and yet giving slower children a chance to accomplish something. Making the same question serve again by adding different figures, or asking quicker children to prove their sum correct, or work out some extra question connected with the sum, or make up a problem for themselves, will serve to keep the brighter minds busy when for any reason it is desired to keep the whole class together in its advance on the new work. To give one question at a time to a large undivided class means to waste the time of the best and to discourage the worst from tackling impossibilities. We need not dwell on the familiar but useful plan of setting questions sufficient to keep the quick children busy, and allowing each child to work on at his own speed.

We have already indicated one or two ways in which we may occasionally use the better brains of the class to aid the slower ones. Children may work in pairs: the children who first finish correctly may have the answers and go round marking and helping the others. Another plan is to use the very best as captains of teams, chosen by themselves. A class of sixty, say, might be divided into ten teams. At times the captain may have the answers and simply supervise his group under the teacher's supervision; at times he may drill his pupils in mental work, more especially in tables, a series of questions and answers prepared by the teacher for the use of captains being most valuable; at times his group may carry through a piece of practical work under his direction. "Matches" between teams may be developed. Obviously such a use of bright children must be strictly limited, but it has its value for themselves as well as for slower pupils.

At this stage some one is sure to ask, "What of examinations?" If a class is doing different work, some perhaps going to higher or lower standards, some never reaching the advanced examples in any of the new work set, some even doing work not done by the rest of the class, what is to happen at the time of tests? Well, surely examinations are to help Arithmetic; not Arithmetic to help examinations! If a child does good work in a lower standard, better suited to his capacity, does he not deserve as much credit as a child who does good work in a higher standard better suited to *his* capacity? Is credit to be given for capacity over which we have little control, or for effort over which we have great control? Then in the ordinary test paper most examiners set some easy work, some work of medium difficulty, and some hard work. Will not the slow pupils, if they have concentrated their energies in covering thoroughly a small part of the ground, do better by succeeding in the easy work than they now do by failing in everything they try? Will not the quick children who have done extra work get recognition of their additional labours in their success with the hard test questions? Would it be really ridiculous if even at times we set easier tests to the C's and harder tests to the A's, so that marks have a closer relationship to effort than to natural capacity? We leave these questions to be answered by the individual teacher or school concerned.

CHAPTER III

THE BALANCE BETWEEN ORAL AND WRITTEN WORK

To achieve the correct balance between oral and written work in Arithmetic is no easy task, partly because of a certain confusion of ideas which arises from a misunderstanding of the terms employed, partly because the right adjustment depends on studying, with the help of pedagogical experiments, some questions still without definite answer. In this chapter we shall do little more than attempt to understand the real bearing of the problem under discussion, feeling inadequate to offer anything approaching a complete solution.

This problem, perhaps, presents itself to us most often in its negative form. How can we prevent our pupils from becoming too dependent on pen and paper in Arithmetic? We see many examples of the disastrous effects of this dependence. Many adults find it extremely hard to perform without paper calculations which it is advantageous to perform in this way. We want to keep a check on the amounts of our purchases when shopping: to discover, only after leaving a shop, that we have received the wrong change, is usually too late. We need to perform calculations over railway time-tables and over the measurement of other quantities. This we indicate as an accepted minimum necessary for all: those who are good at working Arithmetic without paper find their skill useful in many a way not mentioned here. Another disastrous consequence of undue dependence on pen and paper is that when problems have to be solved by written work, so much unnecessary calculation is set down that time is wasted and clearness of style and thought is lost.

But when we pass to the positive aspects of the problem and try to discuss what we should do and what we should leave undone, the terms we employ in our discussion breed additional confusion: *mental*, *oral*, *written*, all three are used in more than one way. From one point of view all Arithmetic is mental Arithmetic. Even the child who sets down each step of his calculations on paper is doing the Arithmetic in his mind, though the more habitual calculations are performed subconsciously: when a slip has to be corrected we realise the mental effort. But by mental Arithmetic we often mean calculations performed in our minds without recording on paper any intermediate results, whether we write down, or say, or simply think the final result. In the ordinary use of language this is the meaning attached to "mental" Arithmetic, and this meaning we shall give to the term in the remainder of this chapter.

We next note that this mental Arithmetic may or may not be "oral" Arithmetic. If the calculation is started by a spoken question, or its final result is recorded in speech, it may be termed "oral"; but how many of the calculations performed by the adult without paper are not of this type! Thinking or reading starts the calculation; the final result is simply retained in the mind or jotted down on paper for future reference. It is true that, in school, far too often the mental Arithmetic is also oral Arithmetic. It is wise to train children to begin mental calculations from the teacher's oral questions, but mental calculations should be begun also from questions written on the blackboard or in the textbook, and from questions raised in the child's mind by some practical problem. It is good sometimes to have the final results recorded in speech, but unless one subdivides a large class into small groups answering their captains, or develops some plan for securing results from nearly all the class, many of the children tend to fall out of such a lesson by the way. Hence, with a large class it is surely simplest to have the answers often recorded in writing; a page in the notebook can be ruled to give five columns for the week's results in mental Arithmetic. The children can write down the result when found or retain it in their minds for a moment until, at the teacher's command, all set it down at once. By some such means it is easy to prevent even a difficult or badly trained class from using paper for the calculation as well as for the answer.

In addition to the fact that mental Arithmetic is not always oral Arithmetic, we must remember that oral Arithmetic is not always mental Arithmetic. We have oral Arithmetic when questions are worked on the blackboard accompanied by class discussion, a type of lesson occurring often when new work is introduced, and frequently in connection with revision and the development of a good style in setting down written work. Sometimes we introduce new subjects by methods purely oral, avoiding any use of either blackboard or paper. We shall do well to remember that in all such oral lessons given to a large class, we run a serious risk of obtaining merely superficial work, and that from only one portion of the class. It is probably wise always to test the results of such oral work by short questions answered on paper at the end of that lesson or before the beginning of the next stage of the work.

Written Arithmetic thus may mean one of a number of things. It may be the record of questions worked without the help of paper: it may be the testing of lessons in which oral discussion has played an important part: it may be the record not only of the final result of a calculation, but of as many intermediate results as must be recorded to relieve the strain on the memory. The last meaning is the one commonly given, and for the remainder of this chapter we shall give this meaning to "written" Arithmetic.

Our problem can now be re-stated in two definite questions, without danger of confusing the terms employed. Is mental or written work the more important and fundamental aspect of Arithmetic? To what extent is it possible to develop ability in mental Arithmetic? And it is precisely to these two questions, and more especially to the second of them, that we feel it impossible to give a definite reply without further experiment.

We can regard the first question from two different points of view. It is partly a matter of age and partly of the stage reached in any branch of the subject. Obviously most types of calculation performed by little children, at least in Standards I. and II., are so little complex that if a child is able to perform them at all, he can usually be trained to record only his result and not his intermediate steps. A few backward pupils with very poor memories may have ultimately to take help by setting down carrying figures and subtractions in division, but the great majority can be trained, if we wish to train them, to record only the final result. We do not mean that in answering such a question as: "5 similar dolls can be bought for 17*s.* 11*d.* Find the cost of each doll," only the answer should be recorded. A proper way of setting down can be practised early—

	s.	*d.*	
5)	17	11	for 5 dolls
	3	7	for 1 doll.

But we do mean that the reduction of 2*s.* to pence, the addition of 11 and the subsequent division by 5 should always be done by mental work. In short, we should give our main attention to developing the child's ability to get the necessary results without any help from paper. At the same time we should realise that it is interesting to the child to learn to record his results in the familiar written form, and that by so doing he may be slowly prepared for the later setting down of complex calculations. But the first part is far more important than the second. Hence, some experienced teachers even advocate no written Arithmetic till Standard IV. is reached. This is a point on which experiment is necessary. Clearly little children want to learn to set down their sums: it is part of their desire to be like grown people. But equally clearly the most important part of their training is learning to do mental Arithmetic. From the point of view of age, then, mental Arithmetic is more important and fundamental than written Arithmetic.

We can get further light on this question by regarding the stage reached in any branch. Perhaps Proportion, as taken in Part II., Chapter XII., furnishes as good an example as any. There the method adopted was to set questions on the new work which could be answered mentally; afterwards to analyse and set down the mental operations which gave a correct answer; finally to generalise these operations so that a method of solving problems too complex for purely mental labour was evolved. Similarly, in teaching younger children reduction, we begin by asking questions answered from mental work, *e.g.* "How many penny buns can I buy with 6*s.* 5*d.*?" When the familiar fact that 12 pence make 1*s.* has been applied naturally in such easy cases, a question involving, say, 17*s.* 9*d.*, will be worked correctly in writing without assistance, although the best form of setting down will not be used. Thus, from the mental working of simple examples, a general method for more complex problems is evolved. A familiar experience gives the same answer to our questions. Have we not often found, especially in Standard IV., which marks the natural transition from emphasis on a simple recording of final results to emphasis on a necessary recording of intermediate results, that children who can work quite stiff questions mentally get them wrong when asked to show their working in writing? The wise folk are not too insistent in such matters: the essential ability is there. Thus, from our experiences in teaching new work at any stage, we should again say that mental Arithmetic is more important and fundamental than written Arithmetic.

To what extent, then, is it possible to develop ability in mental Arithmetic? This question is hard to answer. There are great individual differences, partly due to the fact that some minds are set to work most easily by *seen* stimuli and some by *heard* stimuli, and partly due to variations in memory. Any experimental work done to test these characteristics should be suggestive to us. But it is also true that to a very great extent it is a matter of training how far a child goes in mental Arithmetic. Many adults, quite good at written calculation, are extremely slow and inaccurate in mental work, because they received little or no training when young; this is probably the reason why some of us find it hard to teach mental Arithmetic and are apt to give it too small a place in our plans. Classes of well-trained children may thus be more quick and accurate than their teachers! In such an awkward situation, one in which the author once found herself, nothing avails save careful preparation and a firm resolve not to let mental work slip. Fortunately even adults improve with practice! Some of our heads try to keep us right by labelling certain periods for "oral Arithmetic." We quarrel with the term "oral": we object to such an unnatural divorce of two methods of calculation closely intertwined in any satisfactory scheme of Arithmetic, but we must admit that most of us who ourselves were badly trained in mental Arithmetic, are so apt to neglect it that it is just possible our heads may be justified in their time-table arrangement! Probably a better plan is to set aside the first ten minutes of every lesson for mental Arithmetic. In nine lessons out of ten this can be made to fit in with the rest of the day's work, the exceptions being some lessons on practical work or dealing with the development of a new rule, when a long period is essential. Often the mental work will form the best possible preparation for the remainder of the lesson, and often it will exceed the minimum of ten minutes, even in upper standards. Certainly in schools where pressure is not exerted through the time-table, "ten minutes a day for mental work" is a good safe rule for those of us tempted to slackness in this matter! In the lower standards this allowance is, of course, too scanty.

In teaching almost any part of Arithmetic one finds scope for giving training in mental work. Every one admits that the fundamental questions can be performed mentally with small numbers: we have indicated on p. 43 that addition can be practised mentally to a greater extent than is sometimes realised. Subtraction of two figure numbers is easy and useful if the "making up" method is adopted. Some people say that the mental multiplication of two figure numbers is quite possible with training, and much more rapid than written multiplication. We start with the tens—

$$40 \times 30 = 1200; \ 40 \times 35 = 1200 + 40 \times 5 = 1400;$$
$$46 \times 35 = 40 \times 30 + 40 \times 5 + 30 \times 6 + 6 \times 5;$$

1200, 1400, 1580 and 1620 we count in turn. The way to achieve skill seems to be to hold only one product in the mind and let it go when the next has been added to it. Certainly all can multiply by 25 and $33\frac{1}{3}$ mentally; all can work mentally 102^2 or 99^2. A very suggestive chapter on mental Arithmetic is found in Benchara Branford's *Study of Mathematical Education*, where Bidder, a famous engineer of the last century, advocates teaching mental multiplication up to pairs of three figure numbers! Short methods for money calculations were indicated in Part II. Ability to deal mentally with easy vulgar and decimal fractions and with easy percentages is a necessity. Fractions of £1 and 1*s*. help us to calculate prices. Finding $2\frac{1}{2}\%$ and 5% of sums of money is useful for both interest and discount questions; but after all the main stress should be on the mastery of the fundamental operations up to the limits of individual capacity, and on short ways of money calculation. What the limits are, only experiment can decide.

In conclusion we should note that through written Arithmetic we can give training in mental Arithmetic, if we insist on no unnecessary work being set down and encourage short cuts by every means in our power. Thus rapid working becomes possible and a lucid style engenders clear and purposeful thinking.

CHAPTER IV

THE DANGERS AND DIFFICULTIES OF PRACTICAL WORK

Practical Arithmetic, like Oral Arithmetic, is a somewhat ambiguous term applied to a variety of types of work. From one point of view all Arithmetic should be practical; that is, all Arithmetic should have relation to the demands of life. But in the narrower sense we have "practical" Arithmetic when some form of apparatus beyond pen and paper and text-book is in the hands of a pupil. Before passing to a consideration of the dangers and difficulties of such work, it will help us to analyse rather more carefully the different types of practical lesson. We shall find two main types, introducing apparatus work for entirely different purposes.

In one type of lesson apparatus is introduced as a help towards learning something new, and the success or failure of its introduction must be measured by its effect on skill and accuracy in Arithmetical calculations. In the lowest standards very many lessons of this type are necessary. As we have seen, number ideas and the ideas of the fundamental operations arise through constructive work with a variety of concrete apparatus, all the earliest lessons thus of necessity being practical lessons. We have seen, too, the value of apparatus in the development of the fundamental operations; reference to the chapters on subtraction, multiplication and division will illustrate this point. We find diagrams of immense value in teaching the beginnings of vulgar and decimal fractions and of percentages. Again, in learning to apply this fundamental skill to the needs of daily life in the use of money, weights and measures, we find that familiarity with the actual coins and measures is necessary. The main aim in all such lessons is to produce skill in mental Arithmetic, to teach our pupils how to calculate, to give them possession of a powerful instrument which can be applied by them in life whenever required.

This type of practical lesson obviously must be given just at the time when it is needed. In the earlier stages it will occur much more frequently than in the upper standards; throughout it will be impossible to divide this equally over the weeks of the year. In some weeks the main part of the work may have to be of this type, *e.g.* in the beginning of fractions; in other weeks no reason for apparatus work of this kind can possibly be found. Its success, moreover, must be measured by its effect on skill in calculation. It is not an end in itself: it is a means to a further end. The one great danger to which it is exposed is that of remaining at the level of the concrete or particular, and failing to rise to the abstract and general. For instance, it is most helpful and altogether sound to use apparatus (sticks, counters, tram tickets or flowers) to illustrate division by factors and bring out the relative values of the various remainders. But it is not enough to stop here without formulating the well-known rule for finding the value of the final remainder, and without practising the rule until it is simply mechanical. It is perfectly right to use apparatus (diagrams or coins or lengths of paper) to illustrate the fact that fractional quantities of different forms may be equivalent in value (*e.g.* £$\frac{3}{4}$ and £$\frac{15}{20}$ and £$\frac{60}{80}$), but it is not enough to stop here without formulating the rule for the equivalence of fractions, and without practising the reduction of fractions to their lowest terms until speed and accuracy are attained. The real art in lessons of this type lies in the transition from the concrete to the abstract. At first the concrete predominates; then concrete and abstract go on simultaneously. A little later the work is done without apparatus, but its results are tested by apparatus. Finally, apparatus becomes wholly unnecessary.

No fixed time, however, for the abolition of apparatus can ever be given. Usually the C portion of a class requires it much longer than the A and B portions. We have indicated in Part III., Chapter II., ways by which this difficulty may be met. The inequality of ability in our pupils haunts us throughout lessons of this type, and often makes discipline in such lessons a strain. One help is to decide beforehand the absolutely necessary minimum amount to be expected of the whole class and to have in our minds ideas for using the time of the quicker members while waiting for the others. Sometimes a second piece of similar apparatus work can be set: sometimes quick children may be set to help the slow, this being particularly useful when the work is difficult and we feel the impossibility of helping ten individuals at once; sometimes those who have finished may be set a little problem bearing on the work done or asked to invent a problem for themselves; sometimes they must simply be set to do any written problem or learn something by heart. The main thing is to plan for these odd times work of some kind. Another help with a very large class is to take the lesson in two parts, giving a written test or composition to be done by the half not using the apparatus. We must somehow or other prevent "idle hands," since these usually find mischief!

The aim of the other type of practical lesson is entirely different. Its purpose is to correct the divorce of school Arithmetic from everyday life outside school, to give our pupils opportunities of practising their calculating skill in circumstances comparable to those they will meet in later life, to use children's love of playing at being grown up to incite them to further advances in Arithmetic. The best example of this is found in shopping, as this introduces the use of money, weights and measures, and at every stage makes it easy for the child to realise his need of greater skill in calculation. We have suggested other examples of practical work of this type in the chapters on English and metric weights and measures, on geometry and on the measurement of area and volume. To achieve this aim, the Arithmetical calculations and the practical measurement must be set a-going by the presentation of some problem of interest, for the calculation and the measurement are not ends in themselves, but means to some end desired by the child. The end may be the acquisition of necessary goods by shopping, an acquisition which always requires measurement and calculation; or it may be related to the accomplishment of some journey or the building and decorating of some house; or it may be the making of some object required. The correlation of Arithmetic with handwork of every kind and with geography and science will obviously help here.

More freedom of choice is possible as to the time for giving this type of practical lesson. It is clearly helpful to introduce it as part of the course in that part of the subject with which it is most closely related, so that children feel the worth-whileness of their other Arithmetic lessons; but it is by no means necessary to limit ourselves to this. Any practical lesson which reproduces in the classroom life outside school and is suited in difficulty to the abilities of the children, has a value of its own in our scheme of work. Yet in general its interest is heightened when it is correlated with other work being done about the same time. On the other hand, when the topic of the ordinary Arithmetic lessons is one which is rather dull and whose connection with real life is not obvious, the introduction of such a practical lesson is, by contrast, of real use to all concerned. The main danger of such lessons is that the note of reality tends sometimes to be absent. For instance, a class of children set to measure the volume of wooden cylinders cannot find the same reality in their work as a class set to find out which size of cylindrical cocoa tin is cheapest at the given price. In the invention of interesting problems is scope for both our own and our pupils' ingenuity.

In practical lessons of this second type the inequalities in the class are less troublesome. Often the children work in groups where the quick children render valuable aid as captains. Sometimes each group solves its own problems. Sometimes each group makes its own measurement, and then the whole class solves a problem involving the combined results of the groups. In cases where the children work individually or in pairs and the quick ones forge ahead, it is easy to provide more practical problems to keep all busy. One very useful plan is to write a problem on a slip of paper and on the back of it to write the correct answer, covered by a flap cut, say, from the gummed flap of an envelope. Then the children can work by themselves, coming to the teacher only when completely puzzled, or when they have finished their work correctly and wish to tackle another problem.

Such lessons are, of course, not always perfectly quiet affairs! But, strange to say, the less rigid the discipline of the school, the more quiet they prove to be! Children who are always allowed some safety valve do not need to let off steam as repressed children do. In any case the class can soon be trained to carry through the work without unnecessary disturbance. It is, indeed, part of the preparation for adult life to do Arithmetic in a less artificial atmosphere than that of the usual Arithmetic lesson. Children's real enjoyment of the work makes it easy to check undue noise, as the greatest disappointment for a group is to be set down to do "ordinary" Arithmetic while the others are doing practical work of this type.

In many schools, head masters fix on the time-table periods for Practical Arithmetic. They do this partly because they feel the value of work which helps our pupils to realise that Arithmetic is worth while, partly because Practical Arithmetic is often found a difficult lesson to take and thus tends to be neglected by some of us. The difficulty seems twofold: a difficulty of discipline and a difficulty of preparation. We have indicated above that the difficulty of discipline is mainly a difficulty of preparation, that much of it arises from insufficient planning of work for the quicker pupils. We have hinted, too, at the simplest punishment for classes or individuals that become "rowdy" in practical lessons: they must realise that unless they can display sufficient self-control not to disturb other people, they cannot be allowed much of this interesting work. At the same time we have hinted that sometimes the noise is the fault of the school tone; that where children are living in an atmosphere of repression they must of necessity let off steam when practical lessons give them the opportunity.

Preparation for practical lessons is undoubtedly heavy, and we should expect as much help as possible from our pupils. Some things they cannot prepare for us, but many things they can attend to. A list of our needs for every such lesson should be given in advance to monitors, and they, with co-opted assistants, should provide the necessary articles and put them away afterwards. In smaller classes each individual can tidy for himself. Some of our preparation will diminish as time goes on: we shall be able to use the same problems again for other groups in the class or for a different class. One hint may prove useful. There are times when for the sake of introductory practice in measurement it is desirable to give each child in a class a paper shape of a known size. We have found that the cutting out of these is greatly simplified by the use of those bent-wire paper clips which hold papers together without piercing them. A shape of the required size is placed on the top of from ten to twenty thicknesses of paper and held in position by these clips. The whole is then cut, at each stage a clip being added to hold together parts of the papers which might slip. With sharp scissors fifteen shapes can easily be obtained by one cutting. Thus, even for a class of sixty, four cuttings suffice. To make the pattern shape easily, especially if it involves the drawing of right angles, we have used a sheet of squared paper, as often it is necessary only to draw over lines already there to make the required shape. But, when all is said and done, preparation for such work is not light, and perhaps our chief comfort should be that effort put forth to make such lessons a success is amply repaid by our pupils' quickened interest in their general Arithmetic. Practical Arithmetic, in short, is one aid in abolishing forced labour from our classrooms.

CHAPTER V

THE MARKING AND CORRECTION OF MISTAKES

Our attitude to our pupils' mistakes in Arithmetic we have seen to be one of the most severe tests of our success in teaching. It is fatally easy to become either censorious and harsh, or lax and careless about accuracy. Our patience is often sorely tried, and we have need of all the moral reserves we possess, to combine successfully cheerfulness with firmness, and kindness with zeal for good results. On the one hand, the causes for pupils' bad work may be complicated, ranging as they do from unconquerable physical conditions to such laziness and such lack of goodwill as amount to moral evils. On the other hand, results are often demanded from us which make the holding of an even balance a difficult matter.

The subject we wish to discuss here is the somewhat mechanical business of marking sums right or wrong and having mistakes corrected by the children. A wise or unwise treatment of this business, however, makes a good deal of difference to the progress of a class and to our labour. Our aim is briefly to indicate the variety of methods possible, and make a few comments on the peculiar advantage or disadvantage of each, remembering all the time that with classes of from forty to sixty children ideal methods cannot always be used. We have at times to choose the least of a number of evils, but, even then, we must seek gradually to reach something better. The whole matter is, of course, coloured by the school attitude to wrong sums. Variety, too, must be sought for.

We shall look first at the correction of mistakes which have been by some means or other discovered. Is it true that all mistakes ought to be corrected? Is it true that in every case the child ought to correct his own mistakes?

Clearly correction of mistakes by the individual is most important, but it seems to be a matter of the type of work done whether we shall best achieve this end by insisting on every mistake being corrected as it occurs. In any case, we must see to it that the moral effect of having to get work accurate is not lost. The answers to the above questions depend, too, on our ability to distinguish between mistakes that are fundamental and those that are slips. We can perhaps best deal with the details by considering the different types of work requiring correction.

When mistakes occur in mental tests on tables and short methods, it is seldom of much use to insist on individual correction of the mistake made. By the time it is possible, the child has forgotten the mental process he went through. The normal need seems to be the repeated teaching to the weak section of the class of the operation badly performed, with abundance of drill work gradually increasing in speed and difficulty, use being made either of subdivision of the class or of small groups questioned by their captains. The ordinary class discussion will in general be a sufficient check to the occasional slips made by the better portions of the class. Mistakes in such work are often a matter of speed. Another type of lesson where individual correction of mistakes seems unnecessary is the lesson dealing with new work. Here the conditions of the advance are that mistakes are certain to be made, that those who succeed and those who try to succeed alike merit appreciation. The mistakes made are a great guide to us in our teaching of the new. We alter our teaching to meet the mistakes and then set similar work again, hoping soon for better results. Children always enjoy in such lessons having the number of the class with the question right put on the blackboard each time: to see the number rising and to strive to make it rise quickly, gives distinct pleasure. In such cases to make a mistake is simply part of the day's work!

But when the new idea has been to some extent grasped and the time for drill and practice has come, circumstances are totally different. Where we find odd pupils making fundamental mistakes we must, of course, teach them again the new work and deal tenderly with their errors. But the majority of mistakes will arise from slips of various kinds which may or may not be due to carelessness: experimental determination of the nature of children's mistakes has now been undertaken, and we shall probably, before long, have more definite guidance as to these matters. Nevertheless, almost without exception such slips should be corrected by the pupils who have made them, with or without assistance in discovering their position in the sum. At this stage the right answer must be obtained or the work is not finished. It is easy for us to make this necessary correction anything from an ordinary part of the day's work accepted simply as a necessity by all concerned, to a real punishment for laziness and naughtiness, cutting off the culprit from some much more pleasurable expenditure of his time!

In cases where the questions worked are more problematic and admit of a variety of methods of solution, mistakes may be made in method as well as mistakes which are mere slips. Here the pupil obviously requires assistance from a more successful pupil, or from the teacher. After such help has been received the best plan is either to set the same question again or one similar in method, but embodying different numbers. To pass on without doing this means that we leave a weak link in our chain. The setting of work suited to the various sections of a class clearly makes the giving of the necessary assistance a much easier matter. With a large class it will sometimes be impossible to achieve the whole amount of correction indicated above, but in as far as we fall short, we must realise our definite failure to reach the ideal state of matters.

Obviously the plans adopted to secure the maximum amount of individual correction of mistakes depend on the methods we adopt for the marking of sums as right or wrong. Here we can have much variety, and it seems likely that each method is good in its own time and place. We shall leave till last the question of taking in work and correcting it outside the Arithmetic lesson, partly because an occasional use of this method affords a corrective to the defects of nearly every other plan, but partly because it is the method which makes the greatest demands on the teacher.

One plan which we must use frequently is to go round the class during written work, marking mistakes and giving necessary assistance. If a class has to sit still after every sum until the teacher has marked in this way, this is the worst method of all. But a more sensible use of it is excellent. By no other means can we realise so well the general progress of the work, knowing when to interfere and when to stand aside, when to take a group to the blackboard, when to stop the class to criticise the general work, and when to set harder examples. But we cannot mark the work of every member of a large class by this means; subdivision into sections makes it possible to use this method for any one section on any day. It is one of the best methods for small classes or in small schools where much individual work is carried on. Its unadulterated use, however, sometimes produces queer results; children fail to settle to concentrated labour because of interruption from the teacher, while the class often becomes so dependent on the teacher that it never learns to work on its own initiative. The best section of a class can be badly spoilt in this way and so fall far below the highest of which they are capable. From our own point of view, the drawback to too great a use of it is that it proves very tiring. There seems no reason against an occasional use of the teacher's desk and chair for these purposes. Children can come out in turn, always standing two deep but no more; or those with difficulties may come for part of the lesson and later the teacher go round exercising some general supervision. The relief to ourselves is great and a well-trained class need not suffer. It is probably one of the best methods for Standards VI. and VII. when working revision questions.

Another useful plan, especially in the first stages of any new topic, is to solve on the blackboard with the help of the class the questions set, the pupils marking their work right or wrong. Undoubtedly with a badly trained class, or in a school which lays undue emphasis on getting results, this plan will be abused. Sometimes children may be tempted to alter their work while the discussion continues; sometimes the teacher's desire to know how many went wrong for the sake of a wise selection of the next questions, may lead to children whose sums are wrong saying that they are right. We have ways of checking these evils, but the limitation of the method is that it is not one to be used where much depends on the results. To use it and then blame or punish those who have sums wrong is to court moral disaster in the class; but for general purposes of advance in new work it may be very useful. When circumstances are difficult, the command "pens down" should be given before blackboard work is begun; changing of work can then be checked with comparative ease, and the temptation to dishonesty should not be too great. If a further defence against dishonesty is necessary, children may mark each other's work. In every case the teacher's subsequent supervision of the work either in or out of class helps to keep things right. The fundamental difficulty is the attainment of a good tone in the class; when that has been secured, all else is easy.

In large classes the marking of work done from answers given orally by the teacher is frequently necessary. It may be the marking of the answers to mental tests, or the marking at a certain stage in the lesson of the written work done by some section, so that they may have time to correct mistakes before the close of the lesson. If children mark their own work, it is a help to let them use coloured crayons; these resemble the teacher's blue pencil and avoid the necessity of marking with a pen which can alter figures in the answer. But if punishments or individual marks depend on this, the temptation to dishonesty becomes greater than is desirable. The exchange of books may be used as a half-way house between marking by the individual and marking by the teacher. A large class can be easily trained to do this work well and smartly, on a well-devised mechanical plan. If books are always passed to the child *behind*, the back child in each column passing his book to the front child in the next; if the use of coloured crayon is adopted and the marking is signed with the initials of the person marking; if brief words of command are adopted and if the teacher from time to time carefully supervises the marking, the results will be found to be satisfactory. The children must realise their responsibility in helping the teacher.

Children may be used to mark answers in other ways. A helpful plan is to allow quick children who have done a certain amount of work correctly to have a strip of paper with the answers and go round marking the others' work, sometimes giving assistance and sometimes only marking. Or a list of answers may be pinned up in a convenient place in the room, and children who have finished the work set allowed to go and mark it for themselves. This plan works well with an A section for whom the teacher has little time in one lesson. The children are supposed to continue till the work set is correct. If their books are afterwards inspected, the teacher keeps sufficiently in touch with them to know when assistance is necessary. Another plan is to put on the blackboard the answers to the work set, so that each question may be made right before the next is begun, the teacher's help being asked when the right answer cannot be found. These methods are, of course, better suited to upper standards, but where the atmosphere of the class is right it is surprising how much use may be made of them with highly satisfactory results. In many types of sums children can be trained to prove each answer right before advancing to the next question. Answers are then unnecessary.

When all this has been said and done, corrections outside class remain, we fear, a necessary evil; we have all to try to hold the balance even between letting our classwork suffer through neglect of such supervision of written work, and spending on corrections energy which, applied in another direction, would bring a better return. When a class is new, we shall have frequently to take in written work in order to become acquainted with its Arithmetical powers and the capacities of its individual members. When a class is familiar with us and we with it, sometimes we need give frequent supervision only to those known to be careless or unreliable. We should certainly overlook all books once a fortnight and in general once a week.

This can be quickly done, as detailed marking is not required: it is simply a case of general supervision. Even with a class of sixty, twelve books a day would mean weekly, and six books a day fortnightly supervision. We should also set a test every two or three weeks—perhaps at certain stages, every week; these should be carefully marked, and we should give a full criticism of the work done, arranging to have all mistakes corrected in the following lesson. Such tests are necessary for ourselves as well as for our pupils. If this be done we can make a fairly free use of all other methods of marking work, without much danger of developing dishonesty or carelessness in our classes. These matters of marking are in themselves minor questions, but a wise handling of them is an essential factor in the successful teaching of Arithmetic.

CHAPTER VI

TEXT-BOOKS AND SCHEMES OF WORK

WE can say little in this chapter but what is already obvious to most people, but it may be of use to summarise the main points in connection with these topics, and in conclusion to give typical examples of schemes for our own classes, which allow for the teacher's originality and yet follow the school organisation of the teaching of Arithmetic.

Clearly this is no place to recommend any particular text-book on Arithmetic. Most of us find it useful to have a supply of odd books to provide extra work for the very best members of the class. But, for general work, the familiar little books of examples for the different standards seem best to satisfy the demands of economy and usefulness. To be truly adequate, such books should cater both for the connection of each piece of new work with life, by the setting of suitable examples embodying facts of interest, and for the development of individual skill in manipulation, by giving an abundant supply of more mechanical examples. The old-fashioned books omitted the first; many of the modern books omit the second, but a few can be found which satisfy both needs. So much time is saved by having a satisfactory supply of printed questions that it is a real benefit to have for each class two sets of printed books. Schools using old-fashioned sets of books providing mechanical drill, can add a new set which lays great stress on attractive questions; schools with a set of books which do not give enough of drill work on some topics, can add a set which supplements this deficiency. A school lucky enough to possess one well-balanced set of books need, perhaps, provide only teachers' copies of another set. In girls' schools and boys' schools, books with particularly suitable examples can be introduced; in mixed classes such books should be given to the teacher to be used at his own discretion. One or more sets of cards for each standard may take the place of a second book of examples: remembering our own enjoyment of cards, we are probably right in imagining that children would appreciate their more frequent use nowadays for revision work, especially by the A section of the class.

When this has been said it still remains true that some of the most useful questions must be the product of our own originality or of our pupils'. Often the text-book will suggest the form of a question which will become really "alive" when set with local colouring. It is surprising how good many children are at seeking out questions for themselves. In the commercial and social applications of Arithmetic it is especially important to make the greater part of the questions set, arise from circumstances connected with the particular school or class. The more we know of our children's lives outside, the better we shall become at inventing stimulating problems. Suggestions as to these have been made from time to time throughout Part II.

Some few schools are so much concerned lest we teachers omit this kind of work that they provide no printed text-book at all for the classes. This seems indefensible, as we must then use so much of our energy in hunting for questions and answers for the everyday run of the work, that we have little left for invention and originality. But another reason for providing no text-books is to compel us to take thought for the planning of our work. No text-book in its order of arrangement and in the balance of its different parts suits any one particular class. The text-book must follow the class, rather than the class the text-book. To work straight through a text-book shows a lack of appreciation of the art of teaching Arithmetic and becomes very wearisome to our pupils. Variety is the salt of life to many of us.

How, then, shall we plan our work if we do not follow the text-book? We shall linger for a moment on schemes for a whole school, and then consider the possibility of originality for ourselves. The school scheme in Arithmetic is, of course, a matter for heads of schools. By almost general consent, the scheme now is usually more or less a spiral affair: by this we mean that in each year a little of a variety of work is added. Standard I., for instance, must become sure of number ideas and, to a certain extent, of their notation, and develop the ideas of the fundamental operations and be able to apply them to small numbers. But the apparatus through which this is carried on may easily include the use of simple coins, weights and measures, while the ideas of the simplest fractions naturally introduce themselves. Standards II., III. and IV. deal with developments of the fundamental operations which involve a greater and greater amount of manipulative skill; these operations are practised in applications introducing money, weights, measures and fractions of increasing difficulty, and arise from problems of increasing complexity. Standard V. marks the transition to a more advanced treatment of Arithmetic. Fractions are used in many applications: percentages, proportion, practice, harder areas, harder money calculations all give scope for acquiring mastery over this difficult piece of manipulative work. In Standards VI. and VII. the work taken should be increasingly an application of the fundamental operations to life, while the nature of the applications will vary from district to district and from school to school. The school scheme, then, must allot this work more exactly between the various standards; it must settle the questions of method as to subtraction, decimals and the setting down of multiplication and division, so that children will not have unnecessarily to alter their Arithmetical mechanism as they rise in school, and yet so that the possibility of experiment on new lines is not crowded out; it must arrange for the correlation of Arithmetic with handwork and other subjects. The head of the school must also indicate the general lines on which the work is to be taken, either by his commendation of text-books, or by his tests, or by organised staff discussions. The effect of such organisation on the individual's teaching of Arithmetic may be helpful or the reverse, according to the wisdom of the head of the school, but an effect it must have.

One point on which experiment may well be made is on the method by which the school scheme provides for the teacher's difficulties in teaching the average child. Would it be possible, and if possible good, to have a double scheme for the school? One part of this would embody the work absolutely necessary for even the weaker pupils; each year's work would unfold itself from the preceding year's, the aim being to ensure that weak pupils leave us knowing a few things thoroughly, instead of having an acquaintance with many things so vague that it is of little practical value. The other part of the scheme would embody the work which the school hoped to accomplish with the majority of its pupils, but each part of which is in itself less essential. A good deal of work in this part of the scheme is not affected by other parts of it: nothing fundamental depends on it while it depends on nothing but fundamentals, so that children doing all of it, and part of it, and none of it, are alike ready for the essential work in the succeeding standards. Obviously if this could be done it would be of the greatest help in dealing with the variety of ability before us in our classes.

But when we receive the school scheme for the term or the year from our heads, our work is by no means done. It is true that we cannot often plan the order of the work for a term ahead with a new class, though probably in our third term with it such planning would give us real help. Time spent on making plans quietly and coolly is easily saved later. The method we have found of general use is to plan ahead for a period of from one to four weeks, making our scheme centre round some piece of new work. In addition to the new work, such schemes should provide for variety, for revision, for drill work on fundamental operations, for practical work, for mental work and for tests. The work will sometimes all be centred round the new work, but sometimes it will involve a contrast. In conclusion, we shall apply this idea to making schemes for the teaching of three pieces of new work. These schemes are to be taken as mere suggestions, since all depends on the individual class and the individual teacher. The new work we have selected is long division (Standards III. and IV.), full treatment of vulgar fractions (Standards IV. and V.), areas and volumes in connection with the circle and circular solids (Standards VI. and VII.).

Long division will probably be mastered in not less than three weeks: it may well take four or five. It can be practised not only by itself, but in application to money, weights and measures, and in problems which embody it along with other operations. The drawbacks to teaching it are its real difficulty, the lack of variety in its problems, and its inapplicability to practical work. Since of itself it is good drill in subtraction, multiplication, and division, the weekly drill work would probably best be in addition. Since its problems are monotonous, the revision work had best refer to some piece of work closely connected with life: for instance, revision problems on length. Practical work may be of the nature of revision in measuring length, or some piece of new work may be undertaken which demands little effort in calculation. The illustration of the first and second decimal places by the use of the foot-rule, squared paper and the metric system of length, would be very suitable: see p. 135. Mental work will obviously test weaknesses in multiplication tables, practise mental division by 20, 30, 70, etc., and embody problems on prices and lengths, correlated with the revision of length. On a six-lesson-a-week scheme for four weeks, the allotment of time might be—

	Wk. I.	Wk. II.	Wk. III.	Wk. IV.
Long division .	4	3	4	3
Drill	0	1	0	1
Revision . . .	2	0	2	0
Practical . . .	0	2	0	2
Mental . . .	Every day except in practical work.			

Vulgar fractions will lead to an entirely different scheme. The difficulties here are that the working lessons seldom give practice in involved calculations, that many of the lessons must be practical, and that at the same time the relation of fractions to life is often neglected, and children become wearied. For the sake of clearness we shall imagine that the work has been begun in Standard IV., addition and subtraction having been taught there, but that in Standard V. we are to revise these and advance to multiplication and division. In such circumstances we usually treat fractions as new work, but go over the earlier stages very rapidly. After four weeks the operations should be sufficiently known to be practised on an entirely different piece of new work, say on proportion or areas. The revision work during these four weeks had best be work which provides good problems: for instance, revision problems on money, weights and measures. The drill work can embody all the fundamental operations since none are practised in the new work, or it can select some particularly weak operation and concentrate on it. The practical work will come as part of the new work, except that it will be wise, say twice, to introduce lessons making a definite application of fractions to life. A shopping lesson on harder measurement on the lines of the one taken on p. 98, or the making of some model or plan, using eighths and even sixteenths of an inch, would be suitable. The mental work will give continual easy examples on fractions and particularly develop the use of fractions of 1*s*. and £1, as easy ways of calculating prices. We suggest the following as a possible arrangement of a four-weeks' scheme—

	I.	II.	III.	IV.
Vulgar Fractions .	4	3	4	4
Drill	1	1	1	1
Revision . . .	1	0	1	1
Practical . . .	0	2	0	0
Mental	Every lesson except lessons requiring apparatus.			

Last, we look at suitable Standard VI. and VII. work on measurement of areas and volumes involving the circle. The new work in itself is much more evenly balanced than in the preceding cases. Apparatus is needed to develop the formulæ: practical lessons applying the new to life readily present themselves. The written work on the new of itself revises fractions and may be made sufficiently complex to provide good practice in solving problems. Any general revision taken might deal with miscellaneous questions on areas and volumes, and might be combined with the new in the third week. Drill work can be on anything known to be weak. Mental work does not arise very naturally, and it is probably good to develop something new—perhaps some harder methods for the fundamental operations or for the calculation of prices. Three weeks ought to cover the area of the circle, and the areas and volumes of a cylinder and a cone. We suggest again a possible arrangement—

	I.	II.	III.
Circles, cones, cylinders	5	5	1
Drill work	1	1	1
Revision	0	0	4
Practical	Included in the new work.		
Mental	Every day except when apparatus is necessary.		

These examples have been given to illustrate the amount of variety possible and the value of planning in advance, if we wish to secure a really well-balanced arrangement of the topics set down for the year's or the term's work. Through such schemes, in any school, we can express our own individuality, while at the same time the balance and restraint of our plans unconsciously affect our pupils and contribute valuable elements of stability and order to the atmosphere of our classrooms.

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